Solving Algebraic First Order Differential Equations

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The derivative of any algebraic expression is algebraic. First solve the problem of finding antiderivatives where the solution is a rational expression. Work backwards from the form of the solution to completely characterize those derivatives which can lead to the algebraic solution.

1 Rational Function Differentiation

Let

$$y = \prod_{i=1}^{k} p_i(x)^{n_i}$$

be a rational function of x where the polynomials $p_i(x)$ are squarefree and mutually relatively prime.

The derivative of y is

$$y' = x' \sum_{i=1}^{k} n_i p_i(x)^{n_i - 1} p_i'(x) \prod_{j \neq i} p_j(x)^{n_j} \tag{1}$$

Lemma 1 *The expression $\sum_{i=1}^{k} n_i p_i'(x) \prod_{j \neq i} p_j(x)$ has no factors in common with $p_i(x)$.*

Assume that the expression has a common factor $p_h(x)$. Then

$$p_h(x) \quad \text{divides} \quad \sum_{i=1}^{k} n_i p_i'(x) \prod_{j \neq i} p_j(x)$$

.

Now, $p_h(x)$ divides all terms for $i \neq h$ and since it divides the whole sum, $p_h(x)$ must divide the remaining term $n_h p_h'(x) \prod_{j \neq h} p_j(x)$. But, from the above conditions, $p_h(x)$ does not divide $p_h'(x)$ [$p_h(x)$ is squarefree] and $p_h(x)$ does not divide $p_h(x)$ for $j \neq h$ [relatively prime condition].

2 Rational Function Integration

Now

$$y' = x' \left(\prod_{i=1}^{k} p_i(x)^{n_i - 1} \right) \sum_{i=1}^{k} n_i p_i'(x) \prod_{j \neq i} p_j(x) \tag{2}$$

There are no common factors between the sum and product terms of equation 2 because of the relatively prime condition of equation 1 and because of Lemma 1. Hence, this equation cannot be reduced and is canonical.

Split equation 2 into factors with positive and negative exponents and renumber i to be negative when n_i is negative, giving

$$y' \prod_{-i} p_i(x)^{-n_i+1} = x' \left(\prod_{+i} p_i(x)^{n_i - 1} \right) \sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) \tag{3}$$

Now to integrate equation 3 note that exponents $-n_i+1 > 1$ because $n_i < 0$. Hence $\prod_{-i} p_i(x)^{-n_i+1}$ can factored (easily in fact by squarefree factorization). Now segregate the terms in the sum of equation 2 as well.

$$\begin{aligned} &\sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) = \\ &\sum_{-i} n_i p_i'(x) \prod_{-j \neq -i} p_j(x) \prod_{+k} p_k(x) + \sum_{+i} n_i p_i'(x) \prod_{+j \neq +i} p_j(x) \prod_{-k} p_k(x) \end{aligned}$$

Substituting into equation 3 yields

$$\begin{aligned} &y' \prod_{-i} p_i(x)^{-n_i+1} = \\ &x' \left(\prod_{+i} p_i(x)^{n_i} \right) \sum_{-i} n_i p_i'(x) \prod_{-j \neq -i} p_j(x) + \\ &x' \left(\prod_{+i} p_i(x)^{n_i - 1} \right) \sum_{+i} n_i p_i'(x) \prod_{+j \neq +i} p_j(x) \prod_{-k} p_k(x) \end{aligned} \tag{4}$$

The right side of this equqtion is now grouped into four polynomial terms $AB' + A'B$ where

$$\begin{aligned} A &= \prod_{+i} p_i(x)^{n_i} \\ B' &= \sum_{-i} n_i p_i'(x) \prod_{-j \neq -i} p_j(x) \\ A' &= \left(\prod_{+i} p_i(x)^{n_i - 1} \right) \sum_{+i} n_i p_i'(x) \prod_{+j \neq +i} p_j(x) \\ B &= \prod_{-k} p_k(x) \end{aligned}$$

A is the original numerator and A' it's derivative. B and B' can be derived from the squarefree factorization of the denominator of the integrand. A and A' can be recovered by a kind of long division of the right side of equation 4 by B and B' simultaneously. In addition to subtracting a term times B' subtract the term's derivative times B.

3 A First Order Differential Equation

Starting with equation 1 multiply through by $\prod_{i=1}^{k} p_i(x)$ and replace $\prod_{i=1}^{k} p_i(x)^{n_i}$ on the right side by y.

$$y' \prod_{i=1}^{k} p_i(x) = x' y \sum_{i=1}^{k} n_i p_i'(x) \prod_{j \neq i} p_j(x) \tag{5}$$

By Lemma 1 this cannot be simplified because the two sides have no factor in common. Hence, this form is canonical.

Therefore, given an equation of form $y'q(x) = x'yr(x)$, if it can be put into the form of equation 5, it can be solved as in equation 1. In order to do this we need to factor $q(x)$. This factoring can be seen as the same complexity as the partial fraction decomposition in Risch's algorithm.

Once we have factored $q(x)$, we need to find a set of n_i so that

$$\sum_{i=1}^{k} n_i p_i'(x) \prod_{j \neq i} p_j(x) = r(x)$$

. Now in order for this solution to be unique we need to show that the terms $p_i'(x) \prod_{j \neq i} p_j(x)$ are lineraly independent and hence form the basis for a vector space. Let's assume that they were not independent.

Suppose there existed a set of integers m_i such that

$$\sum_{i=1}^{k} m_i p_i'(x) \prod_{j \neq i} p_j(x) = 0$$

and there exists some $m_i \neq 0$. If only one $m_i \neq 0$ then $p_i'(x) \prod_{j \neq i} p_j(x) = 0$. Since $p_j(x) \neq 0$ then ${p_i}'(x) = 0$. But then $p_i(x)$ would not be a polynomial in x. So then

$$-m_i p_i' \prod_{j \neq i} p_j(x) = \sum_{h \neq i} m_h p_h'(x) \prod_{j \neq h} p_j(x) \tag{6}$$

Now, $p_i(x)$ divides every term on the right side of equation 6 so $p_i(x)$ must also divide $-mp_i'(x) \prod_{j \neq i} p_j(x)$. But, because of squarefree, $p_i(x)$ does not divide $p_i'(x)$ and $p_i(x)$ does not divide $p_j(x)$ when $j \neq i$. Hence, there exists a unique set of coefficients satisfying equation 5.