A Primer on Besov Spaces

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CONNEXIONS

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Chapter 1

Classical measures of smoothness[1]

There exist many different ways of measuring the smoothness of a function f. The most natural one is certainly the order of differentiability, i.e. the maximal index m such that $f^{(m)} = \left(\frac{d}{dx}\right)^m f$ is continuous. To this particular measure of smoothness, we can associate a class of **function spaces**: if I is an interval of $\mathbb{R}$, we denote by $\mathcal{C}^m(I)$ the space of continuous functions which have bounded and continuous derivatives, up to the order m. This space can be equipped with the norm

$$\| f \|_{\mathcal{C}^m(I)} := \sup_{l=0,\cdots,m} \sup_{x\in I} |f^{(l)}(x)|. \tag{1.1}$$

for which it is a Banach space. (That is, the space is a vector space; the norm satisfies the triangle inequality; $\| f \| = 0$ is possible only if $f = 0$; finally, all Cauchy sequences converge: if we have a sequence with entries $f_n \in \mathcal{C}^m(I)$ for which $\| f_n - f_{n'} \|$ can be made arbitrarily small simply by choosing n, n' sufficiently large, then the f_n (and all their derivatives up to the mth) converge uniformly to some function f in $\mathcal{C}^m$ (and its derivatives).

In the case of a multivariate domain $\Omega \in \mathbb{R}^d$, we define $\mathcal{C}^m(\Omega)$ to be the space of continuous functions which have bounded and continuous partial derivatives $\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$, for $|\alpha| := \alpha_1 + \cdots + \alpha_d = 0, \cdots, m$. This space can also be equipped with the norm

$$\| f \|_{\mathcal{C}^m(\Omega)} := \sum_{|\alpha| \le m} \sup_{x\in\Omega} |\partial^\alpha f(x)|, \tag{1.2}$$

for which it is a Banach space.

In many instances, one is somehow interested in measuring smoothness in an average sense: for this purpose it is natural to introduce the **Sobolev spaces**$W^{m,p}(\Omega)$ consisting of all functions $f \in L^p$ with partial derivatives up to order m in L^p. Here p is a fixed index in $[1, +\infty]$. (Recall that $\| f \|_{L^p} = \left[\int_\Omega |f(x)|^p\right]^{\frac{1}{p}}$ if $p < +\infty$ and $\| f \|_{L^\infty} = sup_{x\in\Omega} |f(x)|$.) This space is also a Banach space, when equipped with the norm

$$\| f \|_{W^{m,p}} := \left(\sum_{|\alpha \le m} \| \partial^\alpha f \|_{L^p}^p \right)^{\frac{1}{p}}. \tag{1.3}$$

Note that the norm for $\mathcal{C}^m$ spaces coincides with the $W^{m,\infty}$ norm.

[1] This content is available online at <http://cnx.org/content/m19614/1.3/>.

Chapter 2

Towards fractional smoothness[1]

All the above spaces share the common feature that the regularity index is an integer. In many applications, one is interested to allow fractional order of smoothness, in order to describe the regularity of a function in a more precise way. The question thus arises of **how to fill the gaps between integer smoothness classes**. There are at least two instances where such a generalization is very natural:

- In the case of L^2-Sobolev spaces $H^m := W^{m,2}$ and when $\Omega = \mathbb{R}^d$, we can define an equivalent norm based on the Fourier transform, since by Parseval's formula we have the norm equivalence

$$\| f \|_{H^m}^2 \sim \int_{\mathbb{R}^d} (1+|\omega|)^{2m} \left| \hat{f}(\omega) \right|^2 d\omega. \tag{2.1}$$

  For a non-integer $s \geq 0$, it is thus natural to define the space H^s as the set of all L^2 functions such that

$$\| f \|_{H^s}^2 := \int_{\mathbb{R}^d} (1+|\omega|)^{2s} \left| \hat{f}(\omega) \right|^2 d\omega, \tag{2.2}$$

  is finite.
- In the case of $\mathcal{C}^m$ spaces, we note that $sup_{x\in\Omega}|f(x) - f(x-h)| \leq C|h|$ if $f \in \mathcal{C}^1$ for any $h \in \mathbb{R}^d$ whereas for an arbitrary function $f \in \mathcal{C}^0$, $sup_{x\in\Omega}|f(x) - f(x-h)|$ might go to zero arbitrarily slow as $|h| \to 0$. This motivates the definition of the **Hölder space**$\mathcal{C}^s$, $0 < s < 1$ consisting of those $f \in \mathcal{C}^0$ such that

$$\sup_{x\in\Omega} |f(x) - f(x-h)| \leq C|h|^s. \tag{2.3}$$

  If $m < s < m+1$, a natural definition of $\mathcal{C}^s$ is given by $f \in \mathcal{C}^m$ and $\partial^\alpha f \in \mathcal{C}^{s-m}$, $|\alpha| = m$. It can be proved that this property can also be expressed by

$$\sup_{x\in\Omega} |\Delta_h^n f(x)| \leq C|h|^s, \tag{2.4}$$

  where $n > s$ and Δ_h^n is the n-th order finite difference operator defined recursively by $\Delta_h^1 f(x) = f(x) - f(x-h)$ and $\Delta_h^n f(x) = \Delta_h^1 \left(\Delta_h^{n-1}\right) f(x)$ (for example $\Delta_h^2 f(x) = f(x) - 2f(x-h) + f(x-2h)$). When s is not an integer, the spaces $\mathcal{C}^s$ that we have defined are also denoted as $W^{s,\infty}$. The space $\mathcal{C}^f$ can be equiped with the norm

$$\| f \|_{\mathcal{C}^s(\Omega)} := \| f \|_{L^\infty(\Omega)} + \sup_{h\in\mathbb{R}^d} |h|^{-s} \| \Delta_h^n f \|_{L^\infty(\Omega)}. \tag{2.5}$$

[1] This content is available online at <http://cnx.org/content/m19616/1.3/>.

Let us give two important instances in which the above spaces appear in a natural way. The first is the study of the restriction of a function $f(x_1, \cdots, x_d)$ to a manifold of lower dimension, for example the hyperplane defined by $x_d = 0$. If $g(x_1, \cdots, x_{d-1}) = f(x_1, \cdots, x_{d-1}, 0)$ is such a restriction, then it is known that $f \in H^s\left(\mathbb{R}^d\right.$ for $s > \frac{1}{2}$ implies that $g \in H^{s-\frac{1}{2}}\left(\mathbb{R}^{d-1}\right)$. The second one is the study of the Brownian motion $W(t)$ on an interval I, for which it is known that $W(t)$ is almost surely in $C^{\frac{1}{2}-\epsilon}$ for all $\epsilon > 0$.

Chapter 3

Besov spaces[1]

The definition of "order of smoothness s in L^p" for s non-integer and p different from 2 or ∞ is more subject to arbitrary choices. Among others, one may consider:

- Sobolev spaces $W^{s,p}$ defined (if $m < s < m+1$) by

$$\| f \|_{W^{s,p}} := \left(\| f \|_{W^{m,p}}^p + \sum_{|\alpha|=m} \int_{\Omega\times\Omega} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x-y|^{(s-m)p+d}} dxdy \right)^{\frac{1}{p}} \quad (3.1)$$

  These spaces coincide with those defined by means of Fourier transform when $p=2$ (see [33] for a general treatment).
- Bessel-potential spaces $H^{s,p}$ defined by means of the Fourier transform operator $\mathcal{F}$,

$$\| f \|_{H^{s,p}} = \left(\| f \|_{L^p}^p + \| \mathcal{F}^{-1}(1+|\cdot|^s)\mathcal{F}f \|_{L^p}^p \right)^{\frac{1p}{\cdot}} \quad (3.2)$$

  These spaces coincides with the Sobolev spaces $W^{m,p}$ when m is an integer and $1 < p < +\infty$ (see [3], p.38), but their definition requires that $\Omega = \mathbb{R}^d$ in order to apply the Fourier transform.
- Besov spaces $B_{p,q}^s$, involving an extra parameter q that we define below through finite differences. These spaces include most of those that we have listed so far as particular cases. As we shall see, one of their main interest is that they can be exactly characterized by multiresolution approximation error, as well as from the size properties of the wavelet coefficients.

We define the n-th order L^p modulus of smoothness of f by

$$\omega_n(f,t)_{L^p} = \sup_{|h|\le t} \| \Delta_h^n f \|_{L^p(\Omega_{h,n})}, \quad (3.3)$$

where $\Omega_{h,n} := \{x \in \Omega \,;\, x - kh \in \Omega,\ k = 0, \cdots, n\}$. Here we measure the "size" of $\Delta_h^n f$ in L^p-norm, where we restrict to $L^p(\Omega_{h,n})$ to ensure that all the arguments $x - kh$ occurring in the computation of $\Delta_h^n f(x)$ still live in Ω. For $p, q \ge 1$, $s > 0$, the Besov spaces $B_{p,q}^s$ consists of those functions $f \in L^p$ such that

$$\left(2^{sj}\omega_n\left(f, 2^{-j}\right)_{L^p}\right)_{j\ge0} \in \ell^q. \quad (3.4)$$

Here n is an integer strictly larger than s. A natural norm for such a space is then given by

$$\| f \|_{B_{p,q}^s} := \| f \|_{L^p} + \| \left(2^{sj}\omega_n\left(f, 2^{-j}\right)_{L^p}\right)_{j\ge0} \|_{\ell^q}. \quad (3.5)$$

[1] This content is available online at <http://cnx.org/content/m19611/1.3/>.

If $q = \infty$, the condition (3.4) simply means that $\| \Delta_h^n f \|_{L^p} \leq Ch^{-s}$ for $|h| \leq 1$. For $q < \infty$, the decay condition on $\Delta_h^n f$ is slightly stronger, since we require that the sequence $\left(2^{sj}\omega_n\left(f, 2^{-j}\right)_{L^p}\right)_{j\geq 0}^q$ i be summable. The space $B_{p,q}^s$ also represents "s order of smoothness measured in L^p"; the parameter q allows a finer tuning on the degree of smoothness - one has $B_{p,q_1}^s \subset B_{p,q_2}^s$ if $q_1 \leq q_2$ - but plays a minor role in comparison to s since clearly

$$B_{p,q_1}^{s_1} \subset B_{p,q_2}^{s_2}, \text{ if } s_1 \geq s_2, \tag{3.6}$$

regardless of the values of q_1 and q_2. Roughly speaking, smoothness of order s in L^p is expressed here by the fact that, for n large enough, $\omega_n(f,t)_{L^p}$ goes to 0 like $\mathcal{O}(t^s)$ as $t \to 0$.

Clearly $\mathcal{C}^s = B_{\infty,\infty}^s$ when s is not an integer. It can also be proved that when s is not an integer $W^{s,p} = B_{p,p}^s$. These spaces are different from one another for integer values of s, except when $p = 2$ in which case $H^s = B_{2,2}^s$ for all values of s (see [33], p.38).

Chapter 4

Embeddings[1]

Sobolev, Besov and Bessel-potential spaces satisfy two obvious embedding relations:

- For fixed p (and arbitrary q in the case of Besov spaces), the spaces get larger as s decreases.
- In the case where Ω a bounded domain, for fixed s (and fixed q in the case of Besov spaces), the spaces get larger as p decrease, since $\| f \|_{L^{p_1}} \leq C \| f \|_{L^{p_2}}$ if $p_1 \leq p_2$.

A less trivial type of embedding is known as **Sobolev embedding**. In the case of Sobolev spaces, it states that the continuous embedding

$$W^{s_1,p_1} \subset W^{s_2,p_2} \text{ if } p_1 \leq p_2 \text{ and } s_1 - s_2 \geq d\left(1/p_1 - 1/p_2\right), \tag{4.1}$$

holds except in the case where $p_2 = +\infty$ and s_2 is an integer, for which one needs to assume $s_1 - s_2 > d\left(1/p_1 - 1/p_2\right)$. For example in the univariate case, any H^1 function has also $\mathcal{C}^{1/2}$ smoothness. In the case of Besov spaces the embedding relation are given by

$$B^{s_1}_{p_1,p_1} \subset B^{s_2}_{p_2,p_2} \text{ if } p_1 \leq p_2 \text{ and } s_1 - s_2 \geq d\left(1/p_1 - 1/p_2\right), \tag{4.2}$$

with no other restrictions on the indices $s_1, s_2 \geq 0$. In the case where Ω is a bounded domain, these embedding are compact if and only if the strict inequality $s_1 - s_2 > d\left(1/p_1 - 1/p_2\right)$ holds. The proof of these embeddings can be found in [4] for Sobolev spaces and [34] for Besov spaces.

As an exercise, let us see how these embeddings can be used to derive the range of r such that $B^r_{2,q}\left([0,1]\right)$ can contain discontinuous functions. If $r > 1/2$, then there exists $\epsilon > 0$ such that $r - 2\epsilon > 1/2$; We remark that $B^r_{2,q} \subset B^{r-\epsilon}_{2,q} \subset B^{\epsilon}_{\infty,\infty} = \mathcal{C}^{\epsilon}$, so all functions in $B^r_{2,q}$ are continuous. Therefore only $B^r_{2,q}$ with $r \leq 1/2$ can contain discontinuous functions. In the limiting case $r = 1/2$, a closer inspection reveals that the functions in $B^{1/2}_{2,q}$ are continuous if $q < \infty$, while $B^{1/2}_{2,\infty}$ includes discontinuous functions, such as the characteristic funciton of an interval $[0,a]$ for $0 < a < 1$.

[1] This content is available online at <http://cnx.org/content/m19615/1.5/>.

Chapter 5

Characterization by approximation properties[1]

An important feature of Besov spaces is that they admit equivalent characterization by multiresolution approximation properties and by wavelet decompositions.

Here we use the following standard notation (see [17] or [11] for a general treatment): if f is function we denote by $P_j f$ its projection onto the space V_j, and by $Q_j f = P_{j+1} f - P_j f$ its projection onto the detail space W_j. The multiscale decomposition of f writes

$$f = P_0 f + \sum_{j \geq 0} Q_j f. \tag{5.1}$$

The projectors P_j and Q_j can be further expressed in terms of biorthogonal scaling functions and wavelets bases:

$$P_j f := \sum_{|\lambda| = j} < f, \tilde{\varphi}_\lambda > \varphi_\lambda \ \text{ and } \ Q_j f := \sum_{|\lambda| = j} < f, \tilde{\psi}_\lambda > \psi_\lambda. \tag{5.2}$$

Here we use the simplified notation φ_λ with "$|\lambda| = j$" meaning that the functions are picked at resolution j. In the case where $\Omega = \mathbb{R}^d$, these have the general from $\varphi_\lambda(x) := \varphi_{j,k}(x) := 2^{dj/2} \varphi\left(2^j x - k\right)$, bur for a general domain $\Omega = \mathbb{R}^d$ proper adaptations of these bases need to be done near the boundary. We can therefore write

$$f = \sum d_\lambda \psi_\lambda, \ \ d_\lambda := < f, \tilde{\psi}_\lambda >, \tag{5.3}$$

where we include in this sum the wavelets at all levels $j \geq 0$ and we incorporate the scaling function φ_λ at the first level $|\lambda| = 0$.

Under certain assumptions that we shall discuss below, it is known that the Besov norm $\| f \|_{B^s_{p,q}}$ is equivalent to

$$\| P_0 f \|_{L^p} + \| \left(2^{sj} \| f - P_j f \|_{L^p}\right)_{j \geq 0} \|_{\ell^q}, \tag{5.4}$$

or to

$$\| P_0 f \|_{L^p} + \| \left(2^{sj} \| Q_j f \|_{L^p}\right)_{j \geq 0} \|_{\ell^q} . \tag{5.5}$$

[1] This content is available online at <http://cnx.org/content/m19613/1.3/>.

Using the equivalence $\| Q_j f \|_{L^p} \sim 2^{(d/2-d/p)j} \| (d_\lambda)_{|\lambda|=j} \|_{\ell^p}$ at each level to prove a third equivalent norm in terms of the wavelet coefficients:

$$\| \left(2^{sj} 2^{(d/2-d/p)j} \| (d_\lambda)_{|\lambda|=j} \|_{\ell^p}\right)_{j\geq 0} \|_{\ell^q} . \tag{5.6}$$

These equivalences mean that the modulus of smoothness $\omega_n\big(f, 2^{-j}\big)_{L^p}$ in the definition of $B^s_{p,q}$ can be replaced either by $\| f - P_j f \|_{L^p}$ or by $\| Q_j f \|_{L^p}$. Their validity requires that the spaces V_j satisfy the following two assumptions:

- The V_j must satisfy an approximation property that takes the form of a **direct estimate**

$$\| f - P_j f \|_{L^p} \leq C\omega_n\big(f, 2^{-j}\big)_{L^p}. \tag{5.7}$$

  Such an estimate ensures that a smooth function will have a fast rate of approximation.
- They must also satisfy smoothness properties that takes the form of an **inverse estimate**

$$\omega_n(f_j, t)_{L^p} \leq C\big[min\,\big(1, t2^j\big)\big]^n \| f_j \|_{L^p} \text{ if } f_j \in V_j. \tag{5.8}$$

  Such an estimate takes into account the smoothness of the spaces V_j: it ensures that a function that is approximated at a sufficiently fast rate rate by these spaces should also have some smoothness.

One can show that the direct estimate is satisfied if and only if all polynomials up to order $n-1$ can be written as combinations of the scaling functions φ_λ in V_j, or equivalently if the dual wavelets $\tilde{\psi}_\lambda$ have n vanishing moments. On the other hand, the inverse estimate requires that the scaling functions φ_λ that generates V_j are smooth in the sense of belonging to $W^{n,p}$. Note that the direct estimate immediately implies that the expression (5.4) is less than $\| f \|_{B^s_{p,q}}$. A more refined mechanism, using the inverse estimate (as well as some discrete Hardy inequalities) is used to prove the full equivalence between $\| f \|_{B^s_{p,q}}$ and (5.5) or (5.6). We refer to chapter III in [11] for a detailed proof of these results.

These equivalences show that the convergence rate $N^{-t/d}$ ($N = \dim(V_j)$) can be achieved by the linear multiscale approximation process $f \mapsto P_f$, if and only if the function has roughly "t derivatives in L^p".

Chapter 6

Besov spaces and nonlinear approximation[1]

A natural idea for approximating a function f by wavelets is to retain in the N largest contributions in the norm in which we plan to measure the error. In the case where this norm is L^p, this is given by

$$A_N f := \sum_{\lambda \in E_{N,p}(f)} d_\lambda \psi_\lambda, \tag{6.1}$$

where $E_{N,p}(f)$ is the set of indices of the N largest $\| d_\lambda \psi_\lambda \|_{L^p}$. This set depends on the function f, making this approximation process **nonlinear**. Other instances of nonlinear approximation are discussed in [24].

An important result established in [30] states that $\| f - A_N f \|_{L^p} \sim N^{-r/d}$ is achieved for functions $f \in B^r_{q,q}$ where $1/q = 1/p + r/d$. Note that this relation between p and q corresponds to a critical case of the Sobolev embedding of $B^r_{q,q}$ into L^p. In particular, $B^r_{q,q}$ is not contained in $B^\epsilon_{p,p}$ for any $\epsilon > 0$, so that **no decay rate can be achieved by a linear approximation process** for all the functions f in the space $B^r_{q,q}$. (For **some** functions in $B^r_{q,q}$, which happen to also lie in spaces for which an independent linear approximation theorem can be written, it is of course possible to get a linear approximation rate; the point here is that this is possible only via such additional information.)

Note also that for large values of r, the parameter q given by $1/q = 1/p + r/d$ is smaller than 1. In such a situation the space $B^s_{q,q}$ is not a Banach space any more and is only a quasi-norm (it fails to satisfy the triangle inequality $\| x + y \| \leq \| x \| + \| y \|$). However, this space is still contained in L^1 (by a Sobolev-type embedding) and its characterization by means of wavelets coefficients according to still holds. Letting q go to zero as r goes to infinity allows the presence of singularities in the functions of $B^r_{q,q}$ even when r is large: for example, a function which is piecewise $\mathcal{C}^n$ on an interval except at a finite number of isolated points of discontinuities belongs to all $B^r_{q,q}$ for $q < 1/s$ and $r < n$. This is a particular instance where a non-linear approximation process will perform substantially better than a linear projection.

[1] This content is available online at <http://cnx.org/content/m19612/1.3/>.

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