

State Space Systems

Collection Editor:

Thanos Antoulas

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Collection Editor:

Thanos Antoulas

Authors:

Thanos Antoulas

Elizabeth Chan

Bailey Edwards

JP Slavinsky

Online:

< <http://cnx.org/content/col10143/1.3/> >

C O N N E X I O N S

Rice University, Houston, Texas

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Chapter 1

ELEC 302: State Space Systems Overview¹

1.1 "The Mars Global Surveyor"

"Mars Global Surveyor"



Figure 1.1

¹This content is available online at <<http://cnx.org/content/m2100/2.9/>>.

The Mars Global Surveyor, built by Lockheed Martin and launched by NASA, orbits our nearest planetary neighbor on a mission to map the surface of Mars and catalogue scientific data . The Surveyor spacecraft left Cape Canaveral, Florida aboard a Delta-7925 rocket, and then spent 300 days traveling approximately 750 million kilometers to reach Mars in September of 1997. After the Mars Global Surveyor reached Mars, it used its main rocket engine to lower itself into an elliptical orbit around the planet. The spacecraft then spent the next one and a half years reducing its orbit by using the friction between itself and the atmosphere of Mars to slow down and thus lose 55,000 km of altitude. In March of 1999, the Surveyor spacecraft began its mapping of the Martian surface. The motion of this spacecraft is managed by a propulsion system that consists of a main engine and 8 "attitude-control" thrusters. How do these propulsion devices work together to safely control the movement of the Surveyor spacecraft? In the initial phases of this spacecraft's design, engineers probably asked themselves the following questions to better understand this problem:

- How do we guarantee that the satellite stays in its orbit and doesn't wonder off into space?
- How do we characterize the relationship between the available thrust controls and the position of the spacecraft?
- Can we use the knowledge of the satellite's thruster/position relationship to understand how to efficiently control its movement?
- By observing the satellite's movement, can we better understand of how the dynamics (memory) of the system change with respect to the current and past thruster use?
- Finally, after understanding the dynamics of the system, can we do something to modify them so that the response of the satellite has more desirable properties?

In this course, we will develop ways to answer these questions. In the beginning, we will take a look at linear dynamical systems and determine how to describe their dynamics with a concept known as state. In order to examine these dynamics and see how they form relationships between the inputs and outputs of a system, differential equations and their frequency-domain counterparts will be studied. After setting this foundation, the course material will then focus on concepts found in linear algebra. As many systems have multiple inputs and outputs, it makes sense to use matrices and the tools of linear algebra to deal with the computations involved in describing them.

Once these tools are covered, we can use them along with our knowledge of dynamical systems to analyze the issues mentioned in the example above; specifically, we will examine system stability, controllability, observability, and feedback. With stability, we can see whether the output of a system will remain bounded or whether it will "blow up". This is obviously very useful when thinking about the spacecraft above. As the name implies, controllability of a system tells us whether or not we can control the output of the system without access to the dynamics of the system (i.e. when we can only modify the inputs to the system). The third idea, observability, gives us a method of monitoring the output of a system to determine its state. At the end of the course, we'll see how feedback can use this information about a system's state to alter the system's dynamics in such a way as to improve its properties and response.

To learn more about the Mars Global Surveyor, visit <http://mars.jpl.nasa.gov/mgs/overvu/slides/00.html>². The above image of the MGS was found at <http://mars.jpl.nasa.gov/mgs/images/highres.html>³

²<http://mars.jpl.nasa.gov/mgs/overvu/slides/00.html>

³<http://mars.jpl.nasa.gov/mgs/images/highres.html>

Chapter 2

Matrix Inversion¹

Say that we have the following matrix and that we want to find its determinant.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \quad (2.1)$$

Calculating the determinant of a matrix is a recursive process. Basically, we start by choosing any one row or column. The determinant will then be found with respect to this row or column. What this means is that we will find a sum of the products of this row or column's values and sub-determinants formed by blocking out the row and column of the particular value.

Why is this choice of row or column left to us instead of always being defined as, say, the first row? The reason is that by choosing this row or column wisely, we can sometimes reduce the amount of work we do. For example, if a certain row or column contains a few zeros, choosing it as the row/column that we take the determinant with respect to would be a smart move. As the values of this chosen row or column will be multiplied by sub-determinants of the matrix in question, a value of 0 in one of these products would mean that we have one less matrix whose determinant we need to calculate.

In the case of the matrix above, we'll compute the determinant with respect to the first column. The final equation for the determinant is:

$$\det A = a_{1,1} - 1^{1+1} \det A^{11} + a_{2,1} - 1^{2+1} \det A^{12} + \cdots + a_{n,1} - 1^{n+1} \det A^{1n} \quad (2.2)$$

Here, A^{ij} means the matrix formed by eliminating the i -th column and the j -th row of A .

Let's just look at the first term in final equation for determinant (2.2). It is basically the first element of A 's first column times the determinant of the matrix formed by the elimination of the first row and first column of A . There is also a $(-1)^{r+c}$ term included. This serves to make the signs of all of the terms in the determinant equation fluctuate back and forth. The next term is the same, except that we have moved on to the second element in the first column of A . As this element holds a position in the second row and first column of A , the sub-determinant in this term is obtained by hiding the second row and first column of A .

In a generic 3 x 3 example, we would find the following solution for the determinant:

¹This content is available online at <<http://cnx.org/content/m2118/2.10/>>.

$$\det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = a_{1,1} \det \begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} - a_{2,1} \det \begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} + a_{3,1} \det \begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} \quad (2.3)$$

To find the determinants of the 2 x 2 sub-determinants, we could again apply the rule of the final equation for determinant (2.2), keeping in mind that the determinant of a scalar value is simply that scalar value. However, it is easier to remember the following solution

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \quad (2.4)$$

Example 2.1

To clarify, take the following example of finding the determinant of a numeric 3 x 3 matrix.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ -2 & -2 & 0 \end{pmatrix} \quad (2.5)$$

First we need to choose a row or column to take the determinant with respect to. We notice that the element in the third row and third column is a zero. Knowing that choosing a row or column that contains a zero will reduce our workload, we will choose the third column. Then, by applying final equation for determinant (2.2), we get

$$\det A = 2 \times -1^4 \det \begin{pmatrix} 3 & 1 \\ -2 & -2 \end{pmatrix} + 1 \times -1^5 \det \begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix} + 0 \quad (2.6)$$

Chapter 3

Controllability¹

What do we mean by the term **controllability**? Simply put, we want to know if we can control the state of a system when we only have access to the inputs (i.e. when we can not directly modify the system's state). If we can "steer" a system to a certain state by controlling its inputs, we can then ask ourselves if there is a way to find the most efficient method of making this transformation.

3.1 Developing the Concept of a Controllable Space

Say we have the following system:

$$x' = Ax(t) + Bu(t) \tag{3.1}$$

Example RLC Circuit

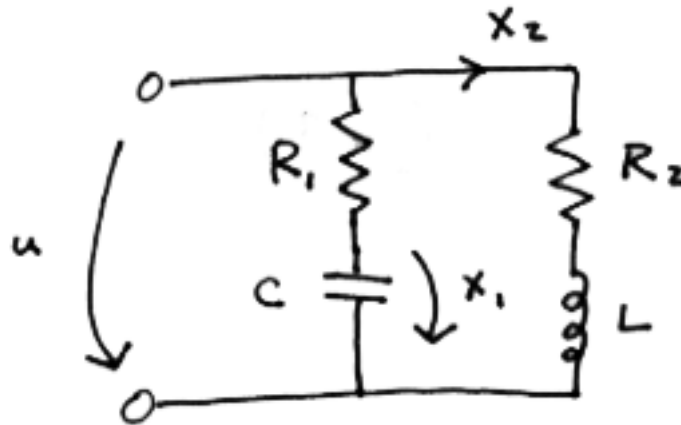


Figure 3.1

¹This content is available online at <http://cnx.org/content/m2120/2.13/>.

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ In this case, an example controllability question could seek to know if there exists an input

u such that: $x(1\text{ms}) = \begin{pmatrix} 10V \\ 1A \end{pmatrix}$

Instead of deriving the general solution for what is called a system's controllable space, X^{contr} , we will simply state it and then give a justification for it.

Formally, X^{contr} is the set of all controllable states. We will define it in terms of a quantity known as the controllability matrix, $C(A, B)$:

$$C(A, B) = \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix} \quad (3.2)$$

The controllable space can then be found by taking the image of this matrix.

$$X^{\text{contr}} = \text{im}(C(A, B)) \quad (3.3)$$

To justify this expression, we begin with the formal matrix equation for a system's state and substitute in the infinite series definition of the matrix exponential. We can then extract the A and B matrices into a larger matrix multiplication.

$$\begin{aligned} x &= \int e^{A(t-\tau)} Bu(\tau) d\tau \\ &= \int \left(I + A(t-\tau) + \frac{A^2}{2}(t-\tau)^2 + \dots \right) Bu(\tau) d\tau \\ &= B \int u(\tau) d\tau + AB \int \frac{t-\tau}{1!} u(\tau) d\tau + A^2B \int \frac{(t-\tau)^2}{2!} u(\tau) d\tau + \dots \\ &= \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix} \begin{pmatrix} \int u(\tau) d\tau \\ \int (t-\tau) u(\tau) d\tau \\ \vdots \\ \int \frac{(t-\tau)^{n-1}}{(n-1)!} u(\tau) d\tau \end{pmatrix} \end{aligned} \quad (3.4)$$

As the second term in the multiplication is dependent on u , it can be thought of as a free variable. Therefore, the set of possible values for x is dependent on the image of first term, which can be seen to be the controllability matrix as defined above.

Continuing the example circuit started above, we can get a better feel for what controllability means. Here are the state equations:

$$\begin{aligned} x_1' &= \frac{-1}{R_1C}x_1 + \frac{1}{R_1C}u \\ x_2' &= -\left(\frac{R_2}{L}x_2\right) + \frac{1}{L}u \end{aligned}$$

Pulling the A and B matrices out of these equations, we can compute the controllability matrix $C(A, B) = \begin{pmatrix} A & AB \end{pmatrix}$. Note that as it is only a second order system, the controllability matrix is only two-dimensional.

$$C(A, B) = \begin{pmatrix} \frac{1}{R_1C} & \frac{-1}{(R_1C)^2} \\ \frac{1}{L} & -\frac{R_2}{L^2} \end{pmatrix}$$

Immediately, we can understand some things about the system by looking at the rank of the C matrix. Let's look at the determinant: $\det C = \frac{1}{LR_1C} \left(-\frac{R_2}{L} + \frac{1}{R_1C} \right)$ If the determinant of the controllability matrix is non-zero, then $X^{\text{contr}} = \text{im}(C) = \mathbb{R}^2$; the system is completely controllable. For this to happen we'd need to

ensure that $\frac{R_2}{L} \neq \frac{1}{R_1 C}$. However, if this inequality is not satisfied and the determinant of the controllability matrix is 0, then we know that it is not full rank. If it is not full rank, then X^{contr} will not span the entire space and the system is not completely controllable. The physical effect here is resonance in the circuit. This reduces our controllability matrix to only one dimension (the two columns are linearly dependent).

$$X^{\text{contr}} = \text{span} \left(\left(\begin{array}{c} \frac{1}{R_1 C} \\ \frac{1}{L} \end{array} \right) \right)$$

Chapter 4

The Concept of State¹

In order to characterize the memory of a dynamical system, we use a concept known as **state**.

NOTE: A system's state is defined as the minimal set of variables evaluated at $t = t_0$ needed to determine the future evolution of the system for $t > t_0$, given the excitation $u(t)$ for $t > t_0$

Example 4.1

We are given the following differential equation describing a system. Note that $u(t) = 0$.

$$\frac{d^1 y(t)}{dt^1} + y(t) = 0 \quad (4.1)$$

Using the Laplace transform techniques described in the module on Linear Systems with Constant Coefficients (Chapter 9), we can find a solution for $y(t)$:

$$y(t) = y(t_0) e^{t_0 - t} \quad (4.2)$$

As we need the information contained in $y(t_0)$ for this solution, $y(t)$ defines the state.

Example 4.2

The differential equation describing an unforced system is:

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{d^1 y(t)}{dt^1} + 2y(t) = 0 \quad (4.3)$$

Finding the $q(s)$ function, we have

$$q(s) = s^2 + 3s + 2 \quad (4.4)$$

The roots of this function are $\lambda_1 = -1$ and $\lambda_2 = -2$. These values are used in the solution to the differential equation as the exponents of the exponential functions:

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} \quad (4.5)$$

where c_1 and c_2 are constants. To determine the values of these constants we would need two equations (with two equations and two unknowns, we can find the unknowns). If we knew $y(0)$

¹This content is available online at <http://cnx.org/content/m2104/2.9/>.

and $\frac{d}{dt}y(0)$ we could find two equations, and we could then solve for $y(t)$. Therefore the system's state, $x(t)$, is

$$x(t) = \begin{pmatrix} y(t) \\ \frac{d^1 y(t)}{dt^1} \end{pmatrix} \quad (4.6)$$

In fact, the state can also be defined as any two non-trivial (i.e. independent) linear combinations of $y(t)$ and $\frac{d}{dt}y(t)$.

NOTE: Basically, a system's state summarizes its entire past. It describes the memory-side of dynamical systems.

Chapter 5

Controllability and Observability Grammians¹

5.1 Controllability Grammian

The finite controllability grammian at time $t < \infty$ is defined as follows.

$$P(t) = \int_0^t e^{A\tau} B B^* e^{A^* \tau} d\tau \quad (5.1)$$

This grammian has two important properties. First, $P(t) = P^*(t) \geq 0$. Secondly, the columns of $P(t)$ span the controllable space, i.e. $\text{im}(P(t)) = \text{im}(C(A, B))$. It can be shown that the state defined by A and B is controllable if, and only if, $P(t)$ is positive definite for some $t > 0$.

Using the controllability grammian, we can determine how to most efficiently take a system from the zero state to a certain state \bar{x} . Given that \bar{x} is in the controllable space, there exists ξ such that

$$\bar{x} = P \left(\frac{\bar{\cdot}}{T} \right) \xi \quad (5.2)$$

for some $\frac{\bar{\cdot}}{T} > 0$. In this case, the minimum energy input required to move the system from zero to \bar{x} is $\bar{u} = B^* e^{A^* \left(\frac{\bar{\cdot}}{T} - t \right)} \bar{\xi}$. If the controllability matrix is invertible, we can use the relation equation between ξ and certain state (5.2) to put \bar{u} in terms of \bar{x} :

$$\bar{u} = B^* e^{A^* \left(\frac{\bar{\cdot}}{T} - t \right)} P^{-1} \left(\frac{\bar{\cdot}}{T} \right) \bar{x} \quad (5.3)$$

In general, this minimal energy is exactly equal to $\bar{\xi}^* P \left(\frac{\bar{\cdot}}{T} \right) \bar{\xi}$. If the system is controllable, then this formula becomes

$$\text{Energy}(\bar{u}) = \bar{x}^* P^{-1} \left(\frac{\bar{\cdot}}{T} \right) \bar{x} \quad (5.4)$$

If you don't want to start at the zero state, the formulas above can still be applied for taking a system at state x_1 to a state x_2 . This holds even if x_1 and x_2 are not controllable; in this case, all that is necessary is for $x_2 - x_1$ to be in the controllable space. (This makes sense if you think of x_1 as being the zero state and x_2 as being the general state we are trying to reach; it is the exact analog of the previous case. Using x_1 and x_2 is just like using 0 and x with an appropriate offset.)

¹This content is available online at <http://cnx.org/content/m2122/2.7/>.

5.2 Observability Grammian

The finite observability grammian at time $t < \infty$ is defined as

$$Q(t) = \int_0^t e^{A^* \tau} C^* C e^{A \tau} d\tau \quad (5.5)$$

Parallel to the finite controllability grammian, the kernel of finite observability grammian is equal to the kernel of the observability matrix. (This relationship holds for positive time only.) $\ker(Q(t)) = \ker(O(C, A))$

Using this grammian, we can find an expression for the energy of the output y at time T caused by the system's initial state x :

$$\text{Energy}(y) = x^* Q(T) x \quad (5.6)$$

5.3 Infinite Grammians

Consider a continuous-time linear system defined, as per normal, by the matrices A , B , C , and D . Assuming that this system is stable (i.e. all of its eigenvalues have negative real parts), both the controllability and observability grammians are defined for $t = \infty$.

$$P = \int_0^\infty e^{A \tau} B B^* e^{A^* \tau} d\tau \quad (5.7)$$

$$Q = \int_0^\infty e^{A^* \tau} C^* C e^{A \tau} d\tau \quad (5.8)$$

These are called the infinite controllability and infinite observability grammians, respectively. These grammians satisfy the linear matrix equations known as the **Lyapunov equations**.

$$AP + PA^* + BB^* = 0 \quad (5.9)$$

$$A^*Q + QA + C^*C = 0 \quad (5.10)$$

In the case of infinite grammians, the equations for minimal energy state transfer and observation energy drop their dependence on time. Assuming stability and complete controllability, the minimal energy required to transfer from zero to state x_c is

$$x_c^* P^{-1} x_c \quad (5.11)$$

Similarly, the largest observation energy produced by the state x_o is obtained for an infinite observation interval and is equal to:

$$x_o^* Q x_o \quad (5.12)$$

Chapter 6

Observability¹

Observability is the tool we use to investigate the internal workings of a system. It lets us use what we know about the input $u(t)$ and the output $y(t)$ to **observe** the state of the system $x(t)$.

To understand this concept let's start off with the basic state-space equations describing a system: $x' = Ax + Bu$ $y = Cx + Du$ If we plug the general solution of the state variable, $x(t)$, into the equation for $y(t)$, we'd find the following familiar time-domain equation:

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau) d\tau + Du(t) \quad (6.1)$$

Without loss of generality, we can assume zero input; this will significantly clarify the following discussion. This assumption can be easily justified. Based on our initial assumption above, the last two terms on the right-hand side of time-domain equation (6.1) are known (because we know $u(t)$). We could simply replace these two terms with some function of t . We'll group them together into the variable $y_0(t)$. By moving $y_0(t)$ to the left-hand side, we see that we can again group $y(t) - y_0(t)$ into another replacement function of t , $\bar{y}(t)$. This result has the same effect as assuming zero input. $\bar{y}(t) = y(t) - y_0(t) = Ce^{At}x(0)$ Given the discussion in the above paragraph, we can now start our examination of observability based on the following formula:

$$y(t) = Ce^{At}x(0) \quad (6.2)$$

The idea behind observability is to find the state of the system based upon its output. We will accomplish this by first finding the initial conditions of the state based upon the system's output. The state equation solution can then use this information to determine the state variable $x(t)$.

base formula (6.2) seems to tell us that as long as we know enough about $y(t)$ we should be able to find $x(0)$. The first question to answer is how much is enough? Since the initial condition of the state $x(0)$ is actually a vector of n elements, we have n unknowns and therefore need n equations to solve the set. Remember that we have complete knowledge of the output $y(t)$. So, to generate these n equations, we can simply take $n - 1$ derivatives of base formula (6.2). Taking these derivatives is relatively straightforward. On the right-hand side, the derivative operator will only act on the matrix exponential term. Each derivative of it will produce a multiplicative term of A . Then, as we're dealing with these derivatives of $y(t)$ at $t = 0$, all of the exponential terms will go to unity ($e^{A0} = 1$).

$$y(0) = Cx(0)$$

$$\frac{d}{dt}y(0) = CAx(0)$$

¹This content is available online at <<http://cnx.org/content/m2121/2.9/>>.

$$\begin{aligned} \frac{d^2}{dt^2}y(0) &= CA^2x(0) \\ &\vdots \\ \frac{d^{n-1}}{dt^{n-1}}y(0) &= CA^{n-1}x(0) \end{aligned}$$

This can be re-expressed in matrix notation.

$$\begin{pmatrix} y(0) \\ \frac{d^1y(0)}{dt^1} \\ \frac{d^2y(0)}{dt^2} \\ \vdots \\ \frac{d^{n-1}y(0)}{dt^{n-1}} \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} x(0)$$

The first term on the right-hand side is known as the observability matrix, $\sigma(C, A)$:

$$\sigma(C, A) = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \quad (6.3)$$

We call the system **completely observable** if the rank of the observability matrix equals n . This guarantees that we'll have enough independent equations to solve for the n components of the state $x(t)$.

Whereas for controllability we talked about the system's controllable space, for observability we will talk about a system's **unobservable** space, X^{unobs} . The unobservable space is found by taking the kernel of the observability matrix. This makes sense because when you multiply a vector in the kernel of the observability matrix by the observability matrix, the result will be 0. The problem is that when we get a zero result for $y(t)$, we cannot say with certainty whether the zero result was caused by $x(t)$ itself being zero or by $x(t)$ being a vector in the nullspace. As we cannot give a definite answer in this case, all of these vectors are said to be unobservable.

One cool thing to note is that the observability and controllability matrices are intimately related:

$$\sigma(C, A)^T = C(A^T, C^T) \quad (6.4)$$

Chapter 7

Diagonalizability¹

A diagonal matrix is one whose elements not on the diagonal are equal to 0. The following matrix is one example.

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

A matrix A is diagonalizable if there exists a matrix $V \in \mathbb{R}^{n \times n}$, $\det V \neq 0$ such that $VAV^{-1} = \Lambda$ is diagonal. In such a case, the diagonal entries of Λ are the eigenvalues of A .

Let's take an eigenvalue decomposition example to work backwards to this result.

Assume that the matrix A has eigenvectors v and w and the respective eigenvalues λ_v and λ_w :

$$Av = \lambda_v v$$

$$Aw = \lambda_w w$$

We can combine these two equations into an equation of matrices:

$$A \begin{pmatrix} v & w \end{pmatrix} = \begin{pmatrix} v & w \end{pmatrix} \begin{pmatrix} \lambda_v & 0 \\ 0 & \lambda_w \end{pmatrix}$$

To simplify this equation, we can replace the eigenvector matrix with V and the eigenvalue matrix with Λ .

$$AV = V\Lambda$$

Now, by multiplying both sides of the equation by V^{-1} , we see the diagonalizability equation discussed above.

$$A = V\Lambda V^{-1} \tag{7.1}$$

When is such a diagonalization possible? The condition is that the algebraic multiplicity equal the geometric multiplicity for each eigenvalue, $\alpha_i = \gamma_i$. This makes sense; basically, we are saying that there are as many eigenvectors as there are eigenvalues. If it were not like this, then the V matrices would not be square, and therefore could not be inverted as is required by the diagonalizability equation (7.1). Remember that the eigenspace associated with a certain eigenvalue λ is given by $\ker(A - \lambda I)$.

¹This content is available online at <<http://cnx.org/content/m2117/2.13/>>.

This concept of diagonalizability will come in handy in different linear algebra manipulations later. We can however, see a time-saving application of it now. If the matrix A is diagonalizable, and we know its eigenvalues λ_i , then we can immediately find the eigenvalues of A^2 :

$$A^2 = (V\Lambda V^{-1})(V\Lambda V^{-1}) = V\Lambda^2 V^{-1}$$

The eigenvalues of A^2 are simply the eigenvalues of A , squared.

Chapter 8

Laplace Example¹

Example 8.1

RLC circuit

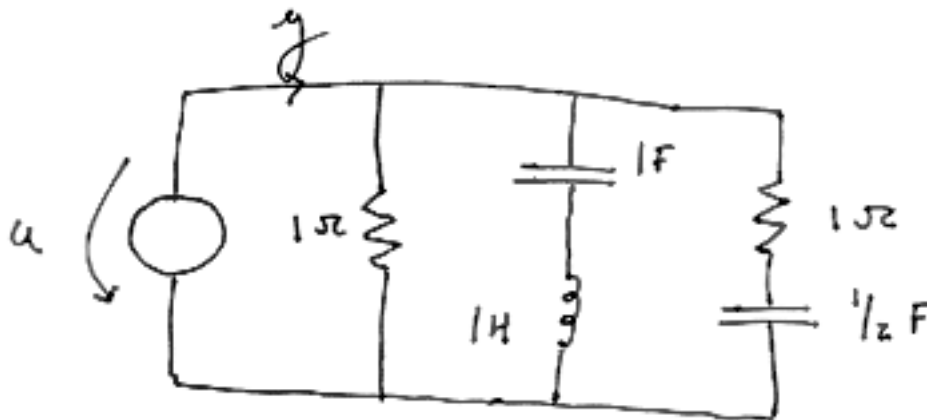


Figure 8.1: RLC circuit

$$y(0^-) = -1 \quad (8.1)$$

$$\frac{d}{dt}y(0^-) = 2 \quad (8.2)$$

$$\frac{d^2}{dt^2}y(0^-) = -4 \quad (8.3)$$

Find the step response for the system above, when $u(t)$ is the input and $y(t)$ is the output (i.e. find $y(t)$ for $u(t) = \text{step}(t)$).

$$Y(s) = \text{Admittance}U(s) \quad (8.4)$$

¹This content is available online at <<http://cnx.org/content/m2109/2.12/>>.

$$\begin{aligned}
\text{Admittance} &= 1 - \text{Impedance} \\
&= 1 + 1 - s + 1 - s + 1 - 1 + 2 - s \\
&= 2s^3 + 3s^2 + 4s + 2 - s^3 + 2s^2 + s + 2
\end{aligned} \tag{8.5}$$

$$Y(s) = (2s^3 + 3s^2 + 4s + 2 - s^3 + 2s^2 + s + 2)U(s) \tag{8.6}$$

With our previous definition of $q(d - d(t))y(t) = p(d - d(t))u(t)$ we can define the Laplace domain equivalents of q and p as:

$$q(s) = s^3 + 2s^2 + s + 2 \tag{8.7}$$

$$p(s) = 2s^3 + 3s^2 + 4s + 2 \tag{8.8}$$

When we multiply $q(s)$ times $Y(s)$, we have to remember to include terms relating to the initial conditions of $y(t)$. We normally think of the Laplace transform of $\frac{d}{dt}y(t)$ as $sY(s)$. However, in reality, the general transform is as follows:

$$\mathcal{L}\left[\frac{d^n y(t)}{dt^n}\right] = s^n Y(s) - s^{n-1}y(0^-) - s^{n-2}\frac{d^1 y(0^-)}{dt^1} - \dots - s\frac{d^{n-2} y(0^-)}{dt^{n-2}} - \frac{d^{n-1} y(0^-)}{dt^{n-1}} \tag{8.9}$$

Therefore, using the initial conditions stated above, we can find the Laplace transforms of the first three derivatives of $y(t)$.

$$\mathcal{L}\left[\frac{d^3 y(t)}{dt^3}\right] = s^3 Y(s) + s^2 - 2s + 4 \tag{8.10}$$

$$\mathcal{L}\left[\frac{d^2 y(t)}{dt^2}\right] = s^2 Y(s) + s - 2 \tag{8.11}$$

$$\mathcal{L}\left[\frac{d^1 y(t)}{dt^1}\right] = sY(s) + 1 \tag{8.12}$$

We can now get a complete s -domain equation relating the output to the input by taking the Laplace transform of $q(d - d(t))y(t) = p(d - d(t))u(t)$. The transform of the right-hand side of this equation is simple as the initial conditions of $y(t)$ do not come into play here. The result is just the product of $p(s)$ and the transform of the step function $(1 - s)$.

The left-hand side is somewhat more complicated because we have to make certain that the initial conditions are accounted for. To accomplish this, we take a linear combination of Laplace transform of the third derivative of $y(t)$ (8.10), Laplace transform of the second derivative of $y(t)$ (8.11), and Laplace transform of the first derivative of $y(t)$ (8.12) according to the polynomial $q(s)$. That is to say, we use the coefficients of the s terms in $q(s)$ to determine how to combine these three equations. We take 1 of Laplace transform of the third derivative of $y(t)$ (8.10) plus 2 of Laplace transform of the second derivative of $y(t)$ (8.11) plus 1 of Laplace transform of the first derivative of $y(t)$ (8.12) plus 2.

When we sum these components, collect the $Y(s)$ terms, and set it equal to the right-hand side, we have:

$$(s^3 + 2s^2 + s + 2)Y(s) + s^2 + 1 = (2s^3 + 3s^2 + 4s + 2)(1 - s) \tag{8.13}$$

Rearranging, we can find the solution to $Y(s)$:

$$Y(s) = (2s^3 + 3s^2 + 4s + 2 - s^3 + 2s^2 + s + 2)(1 - s) - (s^2 + 1 - s^3 + 2s^2 + s + 2) \tag{8.14}$$

This solution can be looked at in two parts. The first term on the right-hand side is the particular (or forced) solution. You can see how it depends on $p(s)$ and $u(s)$. The second term is the homogeneous (or natural) solution. The numerator of this term describes how the initial conditions of the system affect the solution (recall that s^2+1 was the part of the result of the linear combination of Laplace transform of the third derivative of $y(t)$ (8.10), Laplace transform of the second derivative of $y(t)$ (8.11), Laplace transform of the first derivative of $y(t)$ (8.12)). The denominator of the second term is the $q(s)$ polynomial; it serves to describe the system in general.

Chapter 9

Linear Systems with Constant Coefficients¹

9.1 Analyzing Linear Constant Coefficient Systems

Constant coefficient linear systems describe LTI systems and form the beginnings of the study of state-space systems. In general, an n -th order system such as this can be described by the following differential equation, where u and y represent the input and output variables, respectively:

$$\frac{d^n y(t)}{dt^n} + \alpha_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + \alpha_1 \frac{d^1 y(t)}{dt^1} + \alpha_0 y(t) = \beta_m \frac{d^m u(t)}{dt^m} + \beta_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + \beta_1 \frac{d^1 u(t)}{dt^1} + \beta_0 u(t) \quad (9.1)$$

This is a linear differential equation with real-valued constant coefficients α_i and β_j . We can represent this equation with a much more compact notation by thinking of the differential operator as a variable s , where the degree of the derivative corresponds to the power of s . We will define $q(s)$ and $p(s)$ as n -th and m -th order polynomials in s .

$$q(s) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \quad (9.2)$$

$$p(s) = \beta_m s^m + \beta_{m-1} s^{m-1} + \cdots + \beta_1 s + \beta_0 \quad (9.3)$$

If we go ahead and say that $q(s)$ and $p(s)$ will take the differential operator as arguments, we can rewrite (9.1) as

$$q\left(\frac{d}{dt}\right) y(t) = p\left(\frac{d}{dt}\right) u(t) \quad (9.4)$$

Looking at differential equations in terms of polynomials such as $q(s)$ and $p(s)$ right away reminds us the Laplace transform. Using the Laplace transform can often simplify the calculations involved in system analysis.

Example 9.1

By using Laplace domain tools to examine the RLC circuit seen before, the differential equation describing the behavior of this system is easy to deduce.

¹This content is available online at <<http://cnx.org/content/m2103/2.11/>>.

RLC circuit: 2nd order (Impedances Labeled)

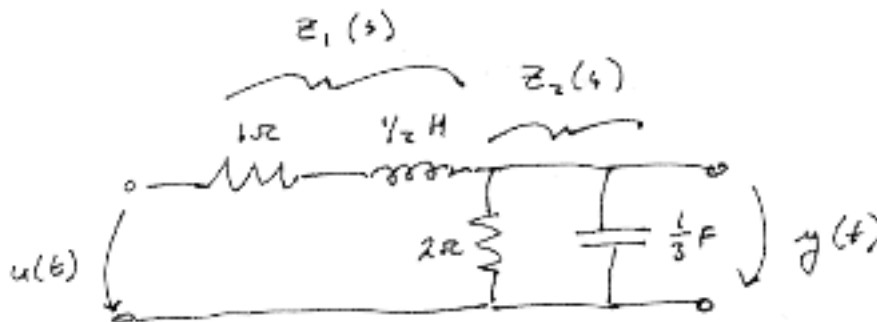


Figure 9.1: The resistor and inductor in series are combined in the impedance $Z_1(s)$, and the resistor and capacitor in parallel form the impedance $Z_2(s)$.

$$\frac{Y(s)}{U(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \quad (9.5)$$

where

$$Z_1(s) = 1 + \frac{s}{2} \quad (9.6)$$

and

$$Z_2(s) = \frac{1}{\frac{1}{2} + \frac{s}{3}} \quad (9.7)$$

This yields

$$Y(s) = \frac{6}{s^2 + \frac{7}{2}s + 9} U(s) \quad (9.8)$$

By multiplying both sides by the denominator of the fraction and taking the inverse Laplace transform, the final differential equation describing the system is determined:

$$\frac{d^2y(t)}{dt^2} + \frac{7}{2} \frac{d^1y(t)}{dt^1} + 9y(t) = 6u(t) \quad (9.9)$$

The results from this example encourage us to apply Laplace techniques to earlier equations in this section. By taking the Laplace transform of (9.4), we find that the transfer function of this system is simply the ratio of the two polynomials $p(s)$ and $q(s)$.

9.1.1 Non-zero Initial Conditions

By saying above that the Laplace transform of $\frac{d}{dt}y(t)$ is $sY(s)$ we are assuming that the system has zero initial conditions. Taking a more general approach, a factor to characterize the initial conditions must be included:

$$\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0^-) \quad (9.10)$$

Note: 0^- here means the instant immediately preceding time $t = 0$; it represents the time right before we start paying attention to the system. Also, some readers may be worried that by using $y(0^-)$ in this equation, we are mixing frequency domain and time domain variables here. However, as $y(0^-)$ is simply a constant, there is no time domain component involved.

Example 9.2

Let's look at an example to see how these non-zero initial conditions come into play. A first order system is described by the following differential equation:

$$\frac{d^1 y(t)}{dt^1} + \alpha_0 y(t) = u(t) \quad (9.11)$$

When we take the Laplace transform of the derivative of $y(t)$, we must remember to include a term that represents the initial values of the system output.

$$sY(s) - y(0^-) + \alpha_0 Y(s) = U(s) \quad (9.12)$$

By combining the $Y(s)$ terms we get

$$(s + \alpha_0)Y(s) - y(0^-) = U(s) \quad (9.13)$$

If we say that $q(s) = s + \alpha_0$ and $p(s) = 1$, and define $r(s)$ as $y(0^-)$, we can rearrange terms to get an expression relating $Y(s)$ to $U(s)$ that takes the initial conditions into account:

$$Y(s) = \frac{p(s)}{q(s)}U(s) + \frac{r(s)}{q(s)} \quad (9.14)$$

What we have here is the Laplace domain solution to a differential equation describing a dynamical system. There are two terms in this solution: one that relies on the input and one that does not. These parts correspond to the particular and homogeneous solutions, respectively. Taking the inverse Laplace transform, we can write (14) as:

$$y(t) = y_{\text{part}}(t) + y_{\text{homo}}(t) \quad (9.15)$$

Here, $y_{\text{part}}(t)$ corresponds to $\frac{p(s)}{q(s)}U(s)$ and $y_{\text{homo}}(t)$ corresponds to $\frac{r(s)}{q(s)}$. This makes a lot of sense. The particular solution (forced response) depends on a combination of $q(s)$, which describes how the system behaves independently, and $p(s)U(s)$, which describes how the system reacts to the input U . The homogeneous solution (natural response) depends on a combination of $q(s)$ and $r(s)$, the latter of which contains information about the initial conditions of the system.

Example 9.3

Let's say that we know the homogeneous solution, $y(t)$, to a differential equation describing a system.

$$y(t) = e^t \quad (9.16)$$

Goal: Using this solution, we want to try and figure out the system's $q\left(\frac{d}{dt}\right)$ function given zero initial conditions.

Solution:

From above, we know that for a homogeneous solution

$$y(t) = \frac{r\left(\frac{d}{dt}\right)}{q\left(\frac{d}{dt}\right)} \quad (9.17)$$

We can clear the denominator by moving the $q\left(\frac{d}{d(t)}\right)$ to the left-hand side. And since we have zero initial conditions, $r\left(\frac{d}{d(t)}\right)$ goes to 0:

$$q\left(\frac{d}{d(t)}\right)y(t) = 0 \quad (9.18)$$

The solution can quickly be determined by inspection because we know that the derivative of e^t is e^t . Therefore a solution of $q\left(\frac{d}{d(t)}\right) = \frac{d}{d(t)} - 1$ would work. However, a more systematic approach will be necessary for more difficult situations. We will investigate this approach here.

Again, we will do our work in the Laplace domain. By equating the Laplace transform of our homogeneous solution with the ratio of $r(s)$ and $q(s)$ as discussed above, we have:

$$\begin{aligned} Y(s) &= \frac{1}{s-1} \\ &= \frac{r(s)}{q(s)} \end{aligned} \quad (9.19)$$

Directly, we can see the solution for $q(s)$: by simply setting the denominators equal to each other, $q(s) = s - 1$. This, of course, is the Laplace transform of the solution of $q\left(\frac{d}{d(t)}\right)$ that we found by inspection above.

Example 9.4

Now that we have the basics down, we'll look at a more complicated example. We are given

$$y(t) = Ae^{at} + Bte^{bt} \quad (9.20)$$

Goal: We would like to find the differential equation whose homogeneous solution is $y(t)$.

Solution:

Again, we take the Laplace transform of $y(t)$, and then combine the two resultant fractions into one ratio of polynomials:

$$\begin{aligned} Y(s) &= \frac{A}{s-a} + \frac{B}{(s-b)^2} \\ &= \frac{A(s-b)^2 + B(s-a)}{(s-a)(s-b)^2} \\ &= \frac{r(s)}{q(s)} \end{aligned} \quad (9.21)$$

Next, we equate the denominators of the last two fractions to find $q(s)$:

$$\begin{aligned} q(s) &= (s-a)(s-b)^2 \\ &= s^3 - (2b+a)s^2 + (b^2 + 2ab)s - ab^2 \end{aligned} \quad (9.22)$$

Recalling the start of this module, multiplying $q(s)$ by $Y(s)$ and taking the inverse Laplace transform will yield the differential equation whose homogeneous solution is $y(t)$:

$$\frac{d^3}{dt^3}y(t) - (2b+a)\frac{d^2y(t)}{dt^2} + (b^2 + 2ab)\frac{d^1y(t)}{dt^1} - ab^2y(t) = 0 \quad (9.23)$$

Chapter 10

Dynamical Systems¹

10.1 "What is a dynamical system?"

When we talk about systems in the most general sense, we are talking about anything that takes in a certain number of inputs and produces a certain number of outputs based on those inputs.

Generalized System

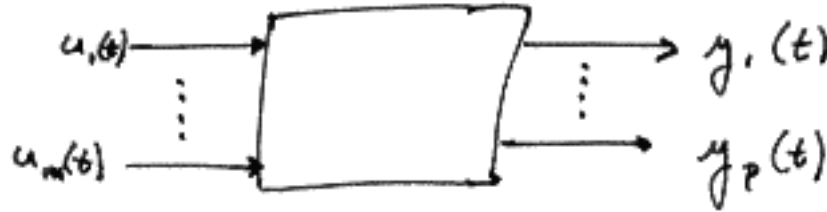


Figure 10.1: Generalized System

In the figure above, the $u(t)$ inputs could be the jets on a satellite and the $y(t)$ outputs could be the gyros describing the "bearing" of the satellite.

There are two basic divisions of systems: static and dynamic. In a static system, the current outputs are based solely on the instantaneous values of the current inputs. An example of a static system is a resistor hooked up to a current source:

¹This content is available online at <http://cnx.org/content/m2101/2.10/>.

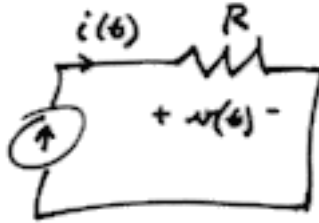
Resistor connected to a current source


Figure 10.2: Resistor connected to a current source

$$V(t) = Ri(t) \quad (10.1)$$

At any given moment, the voltage across the resistor (the output) depends only on the value of the current running through it (the input). The current at any time t is simply multiplied by the constant value describing the resistance R to give the voltage V . Now, let's see what happens if we replace the resistor with a capacitor.

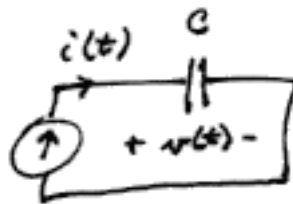
Simple capacitor connected to a current source


Figure 10.3: Simple capacitor connected to a current source

$$I(t) = C \frac{dv(t)}{dt} \quad (10.2)$$

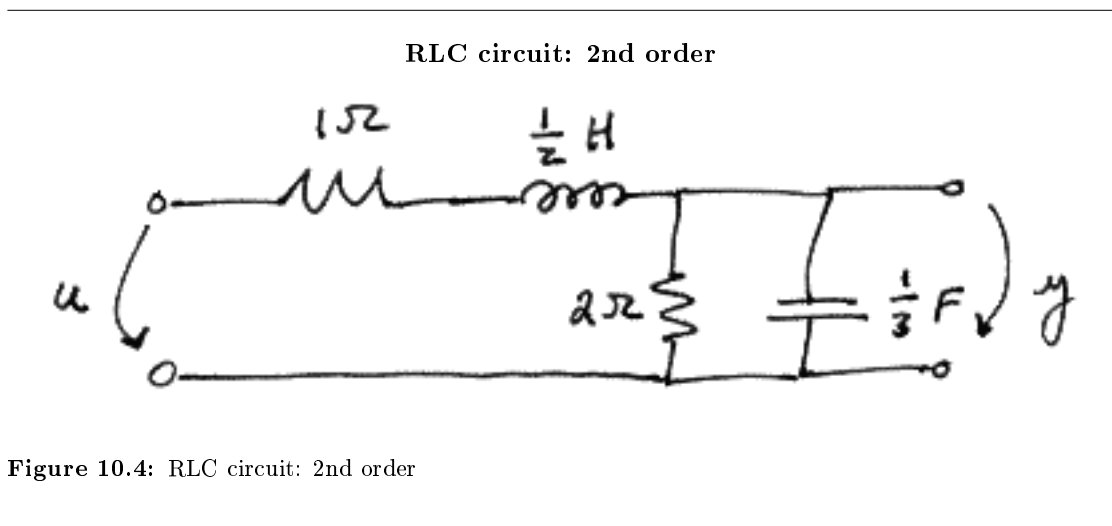
Solving for the voltage in the current voltage relationship above, we have:

$$v(t) - v(t_0) = \frac{1}{C} \int_{t_0}^t i(t) dt \quad (10.3)$$

So in the case of the capacitor, the output voltage depends on the history of the current flowing through it. In a sense, this system has memory. When a system depends on the present and past input, it is said to be a dynamical system.

10.2 "Describing dynamical systems"

As seen in voltage-current relationship of a capacitor, differential equations have memory and can thus be used to describe dynamical systems. Take the following RLC circuit as an example:



In circuits (as well as in other applications), memory elements can be thought of as energy storage elements. In this circuit diagram, there are two energy-storing components: the capacitor and the inductor. Since there are two memory elements, it makes sense that the differential equation describing this system is second order.

$$\frac{d^2y(t)}{dt^2} + \frac{7}{2} \frac{d^1y(t)}{dt^1} + 9y(t) = 6u(t) \quad (10.4)$$

In the most general case of describing a system with differential equations, higher order derivatives of output variables can be described as functions of lower order derivatives of the output variables and some derivatives of the input variables. Note that by saying "function" we make no assumptions about linearity or time-invariance.

By simply rearranging the equation for the RLC circuit above, we can show that that system is in fact covered by this general relationship.

Of course, dynamical systems are not limited to electrical circuits. Any system whose output depends on current and past inputs is a valid dynamical system. Take for example, the following scenario of relating a satellite's position to its inputs thrusters.

10.2.1 "Planar Orbit Satellite"

Example 10.1

Using a simple model of a satellite, we can say that its position is controlled by a radial thruster u_r , which contributes to its vertical motion, and a tangential thruster u_θ which contributes to its motion tangential to its orbit. To simplify the analysis, let's assume that the satellite circles the

earth in a planar orbit, and that its position is described by the distance r from the satellite to the center of the Earth and the angle θ as shown in the figure.

Simple planar orbit satellite example

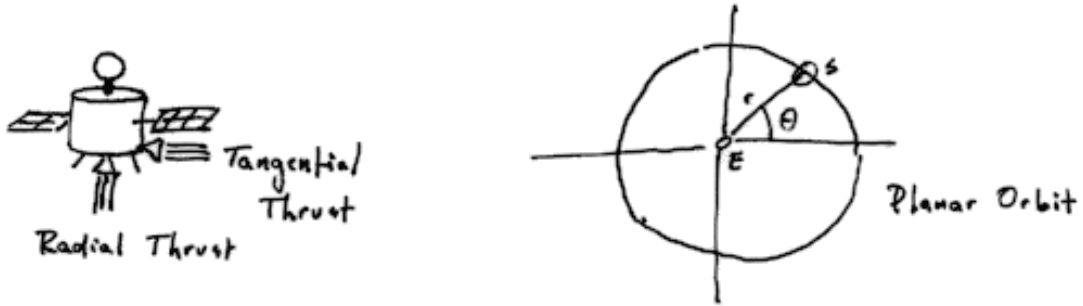


Figure 10.5: Simple planar orbit satellite example

Using the laws of motion, the following set of differential equations can be deduced:

$$\frac{d^2}{dt^2}r(t) - \frac{d^1r(t)}{dt^1}\theta^2 = u_r - \frac{k}{r^2} \quad (10.5)$$

$$2\frac{d^1r(t)}{dt^1}\frac{d^1\theta(t)}{dt^1} + r\frac{d^1\theta(t)}{dt^1} = u_\theta \quad (10.6)$$

Chapter 11

Cayley-Hamilton Theorem¹

The Cayley-Hamilton Theorem states that every matrix satisfies its own characteristic polynomial. Given the following definition of the characteristic polynomial of A ,

$$x_A(\lambda) = \det(\lambda I - A) \quad (11.1)$$

this theorem says that $x_A(A) = 0$. Looking at an expanded form of this definition, let us say that

$$x_A(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$$

Cayley-Hamilton tells us that we can insert the matrix A in place of the eigenvalue variable λ and that the result of this sum will be 0:

$$A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1A + \alpha_0I = 0$$

One important conclusion to be drawn from this theorem is the fact that a matrix taken to a certain power can always be expressed in terms of sums of lower powers of that matrix.

$$A^n = -\alpha_{n-1}A^{n-1} - \cdots - \alpha_1A - \alpha_0I \quad (11.2)$$

Example 11.1

Take the following matrix and its characteristic polynomial.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$x_A(\lambda) = \lambda^2 - 3\lambda + 1$$

Plugging A into the characteristic polynomial, we can find an expression for A^2 in terms of A and the identity matrix:

$$A^2 - 3A + I = 0$$

equation of characteristic polynomial

$$A^2 = 3A - I \quad (11.3)$$

¹This content is available online at <<http://cnx.org/content/m2119/2.13/>>.

To compute A^2 , we could actually perform the matrix multiplication, as below:

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

Or taking equation of characteristic polynomial (11.3: equation of characteristic polynomial) to heart, we can compute (with fewer operations) by scaling the elements of A by 3 and then subtracting 1 from the elements on the diagonal.

$$A^2 = \begin{pmatrix} 6 & 3 \\ 3 & 3 \end{pmatrix} - I = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

Chapter 12

Eigenvalue Decomposition¹

When we apply a matrix to a vector (i.e. multiply them together), the vector is transformed. An interesting question to ask ourselves is whether there are any particular combinations of such a matrix and vector whose result is a new vector that is proportional to the original vector. In math terminology, this question can be posed as follows: if we have a matrix $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$, does there exist a vector $x \in \mathbb{R}^n$ and a scalar $\lambda \in \mathbb{C}$ such that $Ax = \lambda x$? If so, then the complexity of Ax is reduced. It no longer must be thought of as a matrix multiplication; instead, applying A to x has the simple effect of linearly scaling x by some scalar factor λ .

In this situation, where $Ax = \lambda x$, λ is known as an eigenvalue and x is its associated eigenvector. For a certain matrix, each one of its eigenvectors is associated with a particular (though not necessarily unique) eigenvalue. The word "eigen" is German and means "same"; this is appropriate because the vector x after the matrix multiplication is the same as the original vector x , except for the scaling factor. The following two examples give actual possible values for the matrices, vectors, and values discussed in general terms above.

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (12.1)$$

Here, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigenvector and 0 is its associated eigenvalue.

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (12.2)$$

In this second example, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is again the eigenvector but the eigenvalue is now 3.

Now we'd like to develop a method of finding the eigenvalues and eigenvectors of a matrix. We start with what is basically the defining equation behind this whole idea:

$$Ax = \lambda x \quad (12.3)$$

Next, we move the λx term to the left-hand side and factor:

$$(A - \lambda I)x = 0 \quad (12.4)$$

Here's the important rule to remember: there exists $x \neq 0$ satisfying the equation if and only if $\det(A - \lambda I) = 0$. So, to find the eigenvalues, we need to solve this determinant equation.

¹This content is available online at <<http://cnx.org/content/m2116/2.9/>>.

Example 12.1

Given the matrix A , solve for λ in $\det(A - \lambda I) = 0$.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad (12.5)$$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix} \\ &= (\lambda - 2)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \\ &= 0 \end{aligned} \quad (12.6)$$

$$\lambda = \{3, 1\} \quad (12.7)$$

After finding the eigenvalues, we need to find the associated eigenvectors. Looking at the defining equation (12.4), we see that the eigenvector x is annihilated by the matrix $A - \lambda I$. So to solve for the eigenvectors, we simply find the kernel (nullspace) of $A - \lambda I$ using the two eigenvalues we just calculated. If we did this

for the example above, we'd find that the eigenvector associated with $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the eigenvector

associated with $\lambda = 1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

You may be wondering why eigenvalue decomposition is useful. It seems at first glance that it is only helpful in determining the effect a matrix has on a certain small subset of possible vectors (the eigenvectors). However, the benefits become clear when you think about how many other vectors can be looked at from an eigenvalue perspective by decomposing them into components along the available eigenvectors. For instance,

in the above example, let's say we wanted to apply A to the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Instead of doing the matrix

multiply (admittedly not too difficult in this case), the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ could be split into components in the direction of the eigenvalues:

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (12.8)$$

Now, each of these components could be scaled by the appropriate eigenvalue and then added back together to form the net result.

12.1 Multiplicity

Once we have determined the eigenvalues of a particular matrix, we can start to discuss them in terms of their multiplicity. There are two types of eigenvalue multiplicity: algebraic multiplicity and geometric multiplicity.

Definition 12.1: Algebraic Multiplicity

The number of repetitions of a certain eigenvalue. If, for a certain matrix, $\lambda = \{3, 3, 4\}$, then the algebraic multiplicity of 3 would be 2 (as it appears twice) and the algebraic multiplicity of 4 would be 1 (as it appears once). This type of multiplicity is normally represented by the Greek letter α , where $\alpha(\lambda_i)$ represents the algebraic multiplicity of λ_i .

Definition 12.2: Geometric Multiplicity

A particular eigenvalue's geometric multiplicity is defined as the dimension of the nullspace of $\lambda I - A$. This type of multiplicity is normally represented by the Greek letter γ , where $\gamma(\lambda_i)$ represents the geometric multiplicity of λ_i .

12.2 Helpful Facts

Here are some helpful facts about certain special cases of matrices.

12.2.1 Rank

A matrix A is full rank if $\det A \neq 0$. However, if $\lambda = 0$ then $\det(\lambda I - A) = 0$. This tells us that $\det A = 0$. Therefore, if a matrix has at least one eigenvalue equal to 0, then it cannot have full rank. Specifically, for an n -dimensional square matrix:

- When one eigenvalue equals 0, $\text{rank}(A) = n - 1$
- When multiple eigenvalues equal 0 $\text{rank}(A) = n - \gamma(0)$. This property holds even if there are other non-zero eigenvalues

12.2.2 Symmetric Matrices

A symmetric matrix is one whose transpose is equal to itself ($A = A^T$). These matrices (represented by A below) have the following properties:

1. Its eigenvalues are real.
2. Its eigenvectors are orthogonal.
3. They are always diagonalizable.

Chapter 13

Generating Sets and Bases¹

Given $\{x_1, \dots, x_k\} \in \mathbb{R}$, we can define a linear space (vector space) X as

$$\begin{aligned} X &= \text{span}(\{x_1, \dots, x_k\}) \\ &= \left\{ \sum_{i=1}^k \alpha_i x_i \mid \alpha_i \in \mathbb{R} \right\} \end{aligned} \tag{13.1}$$

In this case, $\{x_1, \dots, x_k\}$ form what is known as a **generating set** for the space X . That is to say that any vector in X can be generated by a linear combination of the vectors $\{x_1, \dots, x_k\}$.

If $\{x_1, \dots, x_k\}$ happen to be linearly independent, then they also form a **basis** for the space X . When $\{x_1, \dots, x_k\}$ define a basis for X , k is the dimension of X . A basis is a special subset of a generating set. Every generating set includes a set of basis vectors.

Example 13.1

The following three vectors form a generating set for the linear space \mathbb{R}^2 .

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

It is obvious that these three vectors can be combined to form any other two dimensional vector; in fact, we don't need this many vectors to completely define the space. As these vectors are not linearly independent, we can eliminate one of them. Seeing that x_3 is equal to $x_1 + x_2$, we can get rid of it and say that our **basis** for \mathbb{R}^2 is formed by x_1 and x_2 .

¹This content is available online at <<http://cnx.org/content/m2114/2.9/>>.

Chapter 14

Matrix Exponential¹

Since systems are often represented in terms of matrices and solutions of system equations often make use of the exponential, it makes sense to try and understand how these two concepts can be combined. In many previous applications, we've seen terms like e^{at} come in handy for talking about system behavior. Here, a was always a scalar quantity. However, what would happen if the scalar a was replaced by a matrix A ? The result would be what is known as the **matrix exponential**.

14.1 Definition

Recall the definition of the scalar exponential:

$$e^{at} = 1 + a\frac{t}{1!} + a^2\frac{t^2}{2!} + a^3\frac{t^3}{3!} + \dots \quad (14.1)$$

The definition of the matrix exponential is almost identical:

$$e^{At} = I_n + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \dots \quad (14.2)$$

Where A is $n \times n$ and I_n is the $n \times n$ identity matrix. While it is nice to see the resemblance between these two definitions, applying this infinite series does not turn out to be very efficient in practice. However, it can be useful in certain special cases.

Example 14.1

Compute e^{At} where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We can start by taking powers of A so that we can use the formal definition.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (14.3)$$

¹This content is available online at <<http://cnx.org/content/m2112/2.10/>>.

$$\begin{aligned}
A^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= -I
\end{aligned} \tag{14.4}$$

$$\begin{aligned}
A^3 &= A^2 A \\
&= -A
\end{aligned} \tag{14.5}$$

$$\begin{aligned}
A^4 &= A^2 A^2 \\
&= I
\end{aligned} \tag{14.6}$$

$$\begin{aligned}
A^5 &= A A^2 \\
&= A
\end{aligned} \tag{14.7}$$

$$\begin{aligned}
A^6 &= A^2 A^4 \\
&= -I
\end{aligned} \tag{14.8}$$

And so the pattern goes, giving:

$$A^{4(n-1)+1} = A \tag{14.9}$$

$$A^{4(n-1)+2} = -I \tag{14.10}$$

$$A^{4(n-1)+3} = -A \tag{14.11}$$

$$A^{4(n-1)+4} = I \tag{14.12}$$

If we fill in the terms in the definition of e^{at} , we'll get the following matrix:

$$e^{At} = \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \end{pmatrix} \tag{14.13}$$

We notice that the sums in this matrix look familiar-in fact, they are the Taylor Series expansions of the sinusoids. Therefore, the solution further reduces to:

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \tag{14.14}$$

14.2 General Method

The example above illustrates how the use of the true definition to simplify matrix exponentials might only be easily applied in cases with inherent repetition. There is a more general method involving the Laplace Transform. In particular,

$$\mathcal{L}[e^{At}] = (sI - A)^{-1} \quad (14.15)$$

We can verify that this is true by inserting the formal definition of the matrix exponential:

$$\begin{aligned} \mathcal{L}[e^{At}] &= \mathcal{L}\left[I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \dots\right] \\ &= \frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \dots \\ &= (sI - A)^{-1} \end{aligned} \quad (14.16)$$

The jump between the third and fourth equations here may be a bit hard to believe, but this equality reduces to $I = I$ when both sides are multiplied by $sI - A$. Taking an inverse Laplace of each side of Laplace Transform of the equation (14.15) we find an expression for the matrix exponential:

$$e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) \quad (14.17)$$

Example 14.2

We can do the same example as before, this time using the Laplace-based method.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (14.18)$$

$$\begin{aligned} (sI - A)^{-1} &= \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}^{-1} \\ &= \frac{1}{s^2+1} \begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix} \\ &= \begin{pmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{pmatrix} \end{aligned} \quad (14.19)$$

Taking the inverse laplace of this gives us

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \quad (14.20)$$

14.3 Properties of the Matrix Exponential

In the scalar case, a product of exponentials $e^a e^b$ reduces to a single exponential whose power is the sum of the individual exponents' powers, e^{a+b} . However, in the case of the matrix exponential, this is not true. If A and B are matrices,

$$e^A e^B \neq e^{A+B} \quad (14.21)$$

unless A and B are commutative (i.e. $AB = BA$)

The derivative operates on the matrix exponential the same as it does on the scalar exponential.

$$\begin{aligned}\frac{d}{dt}(e^{At}) &= 0 + A + A^2 \frac{t}{1!} + A^3 \frac{t^2}{2!} + \dots \\ &= A \left(I + A \frac{t}{1!} + A^2 \frac{t^2}{2!} + \dots \right) \\ &= Ae^{At}\end{aligned}\tag{14.22}$$

Chapter 15

Laplace Properties and Transforms¹

15.1 Laplace Properties

$$\begin{aligned}\mathcal{L}[f(t)] &= F(s) \\ &= \int_{0^-}^{\infty} f(t) e^{-(st)} dt\end{aligned}\tag{15.1}$$

Property	Time-domain	Frequency-domain
Linearity	$a f_1(t) + b f_2(t)$	$a F_1(s) + b F_2(s)$
Shifting in s -domain	$e^{s_0 t} f(t)$	$F(s - s_0)$
Time Scaling ($a > 0$)	$f(at)$	$(1 - a) F(s - a)$
Convolution (causal functions)	$f_1(t) * f_2(t)$	$F_1(s) F_2(s)$
Differentiation in Time	$\frac{d}{dt} f(t)$	$sF(s) - f(0^-)$
Differentiation in Freq.	$-(t f(t))$	$\frac{d}{ds} F(s)$
Integration in Time	$\int_{0^-}^t f(\tau) d\tau$	$(1 - s) F(s)$

Figure 15.1

15.2 Unilateral Laplace Transforms

Note: $I(t)$ is a step function.

¹This content is available online at <<http://cnx.org/content/m2110/2.9/>>.

Time-domain	Frequency-domain
$\delta(t)$	1
$I(t)$	$\frac{1}{s}$
$tI(t)$	$\frac{1}{s^2}$
$t^n I(t)$	$\frac{n!}{s^{n+1}}$
$e^{at} I(t)$	$\frac{1}{s-a}$
$te^{at} I(t)$	$\frac{1}{(s-a)^2}$
$\cos(at) I(t)$	$\frac{s}{s^2+a^2}$
$\sin(at) I(t)$	$\frac{a}{s^2+a^2}$

Figure 15.2

Chapter 16

Laplace Domain Solutions to State and Output Equations¹

As always, it is useful to look at the solutions to the state and output equations from a Laplace domain perspective. We start with the general forms of the state and output equations.

$$x'(t) = Ax + Bu \quad (16.1)$$

$$y = Cx + Du \quad (16.2)$$

Let's take a look at the state equation. In the time-domain version of this analysis, we had to use a combination of derivatives and integrals to find the solution $x(t)$. Making an analogy to Laplace-domain equations, we know that derivatives and integrals in time equate to multiplies and divides in frequency. So, we suspect that finding the Laplace-domain solution $X(s)$ might be significantly easier. We will start by taking the Laplace transform of the state equation.

$$\mathcal{L}[x'(t) = Ax + Bu] \quad (16.3)$$

$$sX(s) - x(0) = AX(s) + BU(s) \quad (16.4)$$

If we collect the $X(s)$ terms on the left-hand side, we can come up with

$$(sI - A)X(s) = x(0) + BU(s) \quad (16.5)$$

We see immediately that we the solution for $X(s)$ is staring us in the face. All we have to do is get the $sI - A$ to the other side. Remembering that this term is actually a matrix, we know we can't just divide through by it; instead, we left-multiply both sides of the equation with the inverse of this term. This yields

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \quad (16.6)$$

If we take the time-domain solution for $x(t)$ found before, we can take its Laplace transform and arrive at this same result.

$$\mathcal{L}\left[x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau) d\tau\right] = X(s) \quad (16.7)$$

You can see how this equation would transform into $X(s)$ result equation. We know that the Laplace transform of the matrix exponential is $(sI - A)^{-1}$. The first term is simply this quantity times the scalar

¹This content is available online at <<http://cnx.org/content/m2108/2.8/>>.

$x(0)$. Since the integral term has the form of a convolution, the Laplace-domain equivalent is simple a multiplication.

Finding the Laplace-domain output equation solution is just as easy. Again, we start by taking the Laplace transform of the output equation. This gives us

$$Y(s) = CX(s) + DU(s) \quad (16.8)$$

You might be thinking that this is the end result: we have an expression for $Y(s)$. However, we must remember that we want these solutions to be in terms of known quantities. Namely, these quantities are the initial conditions and the inputs. So what we need to get rid of is the $X(s)$ term. Fortunately, we just found an expression for $X(s)$ in terms of the initial conditions and the inputs. Plugging in that equation into the output equation solution, we get

$$Y(s) = C \left((sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \right) + DU(s) \quad (16.9)$$

When we multiply the C through and collect the $U(s)$ terms, we get a final expression for the output equation solution:

$$Y(s) = C \left((sI - A)^{-1}x(0) \right) + \left(C(sI - A)^{-1}B + D \right) U(s) \quad (16.10)$$

It is interesting to note that the two addends in this equation represent the free response (the initial condition term) and the forced response (the input term) of the system.

Chapter 17

Time Domain State and Output Equations¹

NOTE: x and u are functions of time, and the notation $\frac{d}{dt}(x)$ **implies** $\frac{d}{dt}x(t)$.

Given the state equations and output equations

$$\frac{d}{dt}x(t) = Ax + Bu \quad (17.1)$$

$$y = Cx + Du \quad (17.2)$$

we would like to be able to find solutions for x and y in terms of the initial state of the system and the system's input. To find such solutions, we begin with an examination of a scalar ($n = 1$, $m = 1$) state equation:

$$\frac{d}{dt}x(t) = ax + bu \quad (17.3)$$

If we looked at a special case of this equation, one where the input u was 0, we'd have $\frac{d}{dt}x(t) = ax$. We've seen this many times before; to solve this, we need a function whose derivative is directly proportional to itself. This function is the exponential function. Therefore, in solving the more general case presented by the state equation (17.3), we expect that the exponential function will also come into play.

Starting with the state equation (17.3), we can collect like terms, multiply through by $e^{-(at)}$, and rewrite the left-hand side of the derivative equation (17.4) in terms of the derivative. (We take this last step after noticing that the left-hand side of the derivation equation (17.4) looks like the derivative product rule has already been applied to it.)

$$e^{-(at)} \frac{d^1 x(t)}{dt^1} - ae^{-(at)} x(t) = be^{-(at)} u(t) \quad (17.4)$$

$$\frac{d}{dt} \left(e^{-(at)} x(t) \right) = e^{-(at)} bu(t) \quad (17.5)$$

Since we are searching for $x(t)$ instead of its derivative, we will integrate both sides from t_0 to t .

$$\int_{t_0}^t \frac{d^1 e^{-(at)} x(t)}{dt^1} dt = \int_{t_0}^t e^{-(at)} bu(t) dt \quad (17.6)$$

¹This content is available online at <http://cnx.org/content/m2107/2.9/>.

In the left-hand side of this equation, the integral and the derivative counteract each other and we are left with the difference of the function $e^{-(at)}x(t)$ evaluated at the upper and lower integration limits. To avoid confusion, the variable of integration will be changed from t (which is now a constant limit in the integral) to τ .

$$e^{-(at)}x(t) - e^{-(at_0)}x(t_0) = \int_{t_0}^t e^{-(a\tau)}bu(\tau) d\tau \quad (17.7)$$

We now move the $x(t_0)$ term to the other side and divide through by $e^{-(at)}$. This leaves us with a solution for the state variable $x(t)$ in the scalar case:

$$x(t) = e^{a(t-t_0)}x(t_0) + \int_{t_0}^t e^{a(t-\tau)}bu(\tau) d\tau \quad (17.8)$$

What happens if we let t_0 go to $-\infty$? The first term on the right-hand side will go to zero since $x(-\infty) = 0$. Then, if we say that $h(t) = e^{at}b$, the second term can be rewritten as

$$\int_{-\infty}^t h(t-\tau)u(\tau) d\tau \quad (17.9)$$

This is the convolution equation $h * u$.

For the scalar case, the solution to the output equation $y(t)$ has the same basic form as the solution to the state equation:

$$y(t) = ce^{a(t-t_0)}x(t_0) + \int_{t_0}^t ce^{a(t-\tau)}bu(\tau) d\tau + d(u(t)) \quad (17.10)$$

Again, we can see the convolution in the second term.

The general matrix forms of the solutions for the state and output equations follow the same pattern. The only differences are that we use the matrix exponential instead of the scalar exponential, and that we use the matrices A , B , C , and D instead of the scalars a , b , c , and d .

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau) d\tau \quad (17.11)$$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau) d\tau + Du(t) \quad (17.12)$$

The convolution term is easy to see in the solution for $x(t)$. However, it takes a little regrouping to find it in the solution for $y(t)$. If we pull the D matrix into the integral (by multiplying it by the impulse function, δ , we once again have the integral of a function of $t - \tau$ being multiplied by the input (convolution).

$$y(t) = Ce^{At}x(0) + \int_0^t \left(Ce^{A(t-\tau)}B + D\delta(t-\tau) \right) u(\tau) d\tau \quad (17.13)$$

Chapter 18

Matrix Inversion¹

Let's say we have the square $n \times n$ matrix A composed of real numbers. By "square", we mean it has the same number of rows as columns.

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \quad (18.1)$$

The subscripts of the real numbers in this matrix denote the row and column numbers, respectively (i.e. $a_{1,2}$ holds the position at the intersection of the first row and the second column).

We will denote the inverse of this matrix as A^{-1} . A matrix inverse has the property that when it is multiplied by the original matrix (on the left or on the right), the result will be the identity matrix.

$$\begin{aligned} AA^{-1} &= A^{-1}A \\ &= I \end{aligned} \quad (18.2)$$

To compute the inverse of A , two steps are required. Both involve taking determinants ($\det A$) of matrices. The first step is to find the adjoint (A^H) of the matrix A . It is computed as follows:

$$A^H = \begin{pmatrix} \alpha_{1,1} & \dots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \dots & \alpha_{n,n} \end{pmatrix} \quad (18.3)$$

$$\alpha_{i,j} = -1^{i+j} \det A^{ij} \quad (18.4)$$

where A^{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by eliminating its i -th column and j -th row. Note that we are not eliminating the i -th row and j -th column as you might expect.

To finish the process of determining the inverse, simply divide the adjoint by the determinant of the original matrix A .

$$A^{-1} = \frac{1}{\det A} A^H \quad (18.5)$$

¹This content is available online at <<http://cnx.org/content/m2113/2.10/>>.

Example 18.1

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (18.6)$$

Find the inverse of the above matrix.

The first step is to compute the terms in the adjoint matrix:

$$\alpha_{1,1} = -1^2 d \quad (18.7)$$

$$\alpha_{1,2} = -1^3 b \quad (18.8)$$

$$\alpha_{2,1} = -1^3 c \quad (18.9)$$

$$\alpha_{2,2} = -1^4 a \quad (18.10)$$

Therefore,

$$A^H = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (18.11)$$

We then compute the determinant to be $ad - bc$. Dividing through by this quantity yields the inverse of A :

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (18.12)$$

Chapter 19

Matrix Representation of Systems¹

19.1 State Equations

Knowing that a system's state describes its dynamics, or memory, it is also useful to examine how the state of a system evolves over time. A system's state will vary based on the current values of the state as well as the inputs to the system:

$$\frac{d^1 y(t)}{dt^1} + y(t) = 0 \quad (19.1)$$

Looking at an example will help to see why calculating the time-varying behavior of the state is important.

Example 19.1

A system is described by the following differential equation:

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{d^2 y(t)}{dt^2} + 2y(t) = 0 \quad (19.2)$$

The state of this system is

$$\begin{aligned} x(t) &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &= \begin{pmatrix} y(t) \\ \frac{d^1 y(t)}{dt^1} \end{pmatrix} \end{aligned} \quad (19.3)$$

The state $x(t)$ (a vector) is composed of two state variables $x_1(t)$ and $x_2(t)$. We would like to be able to talk about the time-varying state in terms of these state variables. That is, we'd like an expression where $\frac{d}{dt}x(t)$ can be written in terms of $x_1(t)$ and $x_2(t)$. From state equation (19.3) above, we see that $\frac{d}{dt}x_1(t)$ simply equals $\frac{d}{dt}y(t)$. In the same equation we also notice that $\frac{d}{dt}y(t)$ equals $\frac{d}{dt}x_2(t)$. Therefore, the derivative of the first state variable exactly equals the second state variable.

$$\begin{aligned} \frac{d}{dt}x_1(t) &= \frac{d}{dt}y(t) \\ &= x_2(t) \end{aligned} \quad (19.4)$$

We can follow the same process for $x_2(t)$. Again from state equation (19.3), we see that the first derivative of $x_2(t)$ equals the second derivative of $y(t)$. At this stage, we can bring in information

¹This content is available online at <<http://cnx.org/content/m2106/2.11/>>.

from the system's differential equation. That equation (the first one in this example) also contains the second derivative of $y(t)$. If we solve for it we get

$$\frac{d^2}{dt^2}y(t) = -3\frac{d^1y(t)}{dt^1} - 2y(t) + u(t) \quad (19.5)$$

We already know that $\frac{d}{dt}y(t)$ equals $x_2(t)$ and that $y(t)$ equals $x_1(t)$. Putting all of this together, we can get an expression for $\frac{d}{dt}x_2(t)$ in terms of the state variables and the input variable.

$$\begin{aligned} \frac{d}{dt}x_2(t) &= \frac{d^2}{dt^2}y(t) \\ &= -3\frac{d^1y(t)}{dt^1} - 2y(t) + u(t) \\ &= -3x_2(t) - 2x_1(t) + u(t) \end{aligned} \quad (19.6)$$

The important thing to notice here is that by looking at the time-varying behavior of the state, we have been able to reduce the complexity of the problem. Instead of one second-order differential equation we now have two first-order differential equations.

Think about a case where we might have 5, 10, or even 20 state variables. In such an instance, it would be difficult to work with so many equations. For this reason (and in order to have a more compact notation), we represent these state variable equations in terms of matrices. The set of equations above can be written as:

$$\frac{d}{dt} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \quad (19.7)$$

By letting $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we can rewrite this equation as:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) \quad (19.8)$$

This is called a **state equation**.

State equations are **always** first-order differential equations. All of the dynamics and memory of the system are characterized in the state equations. In general, in a system with n state variables and m inputs, A is $n \times n$, $x(t)$ is $n \times 1$, B is $n \times m$, and $u(t)$ is $m \times 1$.

State Equation Matrices

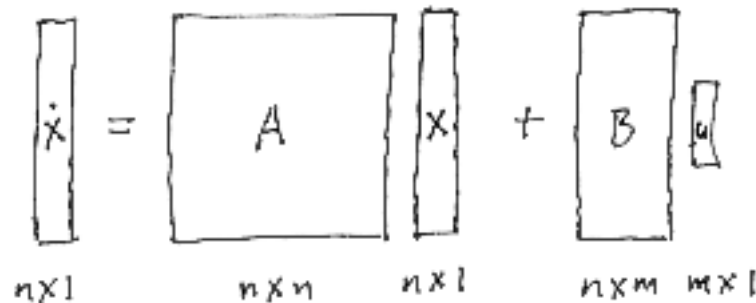


Figure 19.1: State Equation Matrices

19.2 Output Equations

Now that we've seen how to examine a system with respect to its state equations, we can move on to equations defining the relationships between the outputs of the system and the state and input variables. The outputs of a system can be written as sums of linear combinations of state variables and input variables. If in the example above the output $y(t)$ depended only on the first state variable, we could write $y(t)$ in matrix form:

$$\begin{aligned} y(t) &= x_1(t) \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \end{aligned} \quad (19.9)$$

More generally, we can express the output (or outputs) as:

$$y(t) = Ax(t) + Du(t) \quad (19.10)$$

In a system with m inputs, n state variables, and p outputs, $y(t)$ is $p \times 1$, C is $p \times n$, $x(t)$ is $n \times 1$, D is $p \times m$, and $u(t)$ is $m \times 1$. Output equations are only algebraic equations; there are no differential equations and therefore, there is no memory component.

If we assume that $m = p = 1$ and $D = 0$, we can eliminate $x(t)$ in a combination of the state equations and output equations to get the input/output relation $q(d - d(t))y(t) = p(d - d(t))u(t)$. Here the degree of q equals the degree of p .

Example 19.2

Let's develop state and output equations for the following circuit diagram:

Example Circuit 1

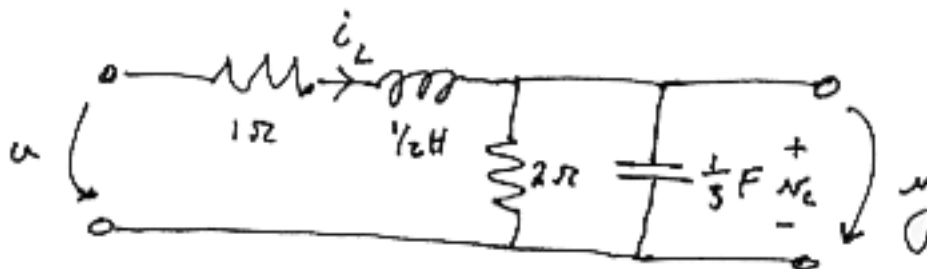


Figure 19.2: Example Circuit 1

There are two energy-storage elements in this diagram: the inductor and the capacitor. As we know that energy-storage elements give systems memory, it makes sense that the state variables should be the current i_L flowing through the inductor and the voltage v_C across the capacitor. By using Kirchoff's laws around the left and center loops, respectively, we can find the following two equations:

$$u = i_L - 1 \frac{di_L}{dt} + v_C \quad (19.11)$$

$$i_L = v_C - 2 - 1 \frac{dv_C}{dt} \quad (19.12)$$

These equations can easily be rearranged to have the derivatives on the left-hand side equaling linear combinations of state variables and inputs on the right. These are the state equations. The figure also quickly tells us that the output y is equal to the voltage across the capacitor, v_C .

We can now rewrite the state and output equations in matrix form:

$$\begin{pmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 3 & -3-2 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (19.13)$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} u \quad (19.14)$$

19.3 Compact System Notation

We now introduce one more simple way to simplify the representation of systems. Basically, to better use the tools of linear algebra, we will put all four of the matrices from the state and output equations (i.e., A , B , C , and D) into one large partitioned matrix:

Compact System Matrix Notation

The diagram shows a large square matrix enclosed in square brackets. A vertical line and a horizontal line intersect at the center, dividing the matrix into four quadrants. The top-left quadrant is labeled 'A', the top-right 'B', the bottom-left 'C', and the bottom-right 'D'. Above the matrix, the letter 'n' is written in blue above the top-left corner, and 'm' is written in blue above the top-right corner. To the left of the matrix, the letter 'n' is written in blue vertically, and the letter 'P' is written in blue vertically below it.

Figure 19.3: Compact System Matrix Notation

Example 19.3

In this example we'll find the state and output equations for the following circuit, as well as represent the system using the compact notation described above.

Example Circuit 2

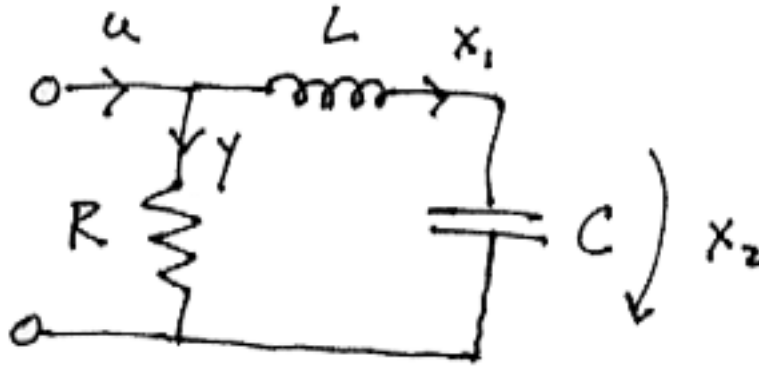


Figure 19.4: Example Circuit 2

Here, u and y are the input and output currents, respectively. x_1 and x_2 are the state variables. Using Kirchoff's laws and the i - v relation of a capacitor, we can find the following three equations:

$$u = y + x_1 \quad (19.15)$$

$$x_1 = A \frac{dx_2}{dt} \quad (19.16)$$

$$Ry = \mathcal{L} \left[\frac{dx_1}{dt} \right] + x_2 \quad (19.17)$$

Through simple rearranging and substitution of the terms, we find the state and output equations:

State equations:

$$\frac{d}{dt}(x_1) = -1x_2 - R(u - x_1) \quad (19.18)$$

$$\frac{d}{dt}(x_2) = (1 - A)x_1 \quad (19.19)$$

Output equation:

$$y = -x_1 + u \quad (19.20)$$

This equations can be more compactly written as:

$$\begin{pmatrix} A & B \\ A & D \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} (-R) - L & -1 - L \\ 1 - A & 0 \end{pmatrix} & \begin{pmatrix} R - L \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix} \quad (19.21)$$

Example 19.4

The simple oscillator is defined by the following differential equation:

$$\frac{d^2 y}{dt^2} + y = u \quad (19.22)$$

The states are $x_1 = y$ (which is also the output equation) and $x_2 = \frac{d}{dt}(y)$. These can be rewritten in state equation form as:

$$\frac{d}{dt}(x_1) = x_2 \quad (19.23)$$

$$\frac{d}{dt}(x_2) = -x_1 + u \quad (19.24)$$

The compact matrix notation is:

$$\begin{pmatrix} A & B \\ A & D \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \quad (19.25)$$

Chapter 20

Defining the State for an n-th Order Differential Equation¹

Consider the n-th order linear differential equation:

$$q(s)y(t) = p(s)u(t) \quad (20.1)$$

where $s = \frac{d}{dt}$ and where

$$q(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1s + \alpha_0 \quad (20.2)$$

$$p(s) = \beta_{n-1}s^{n-1} + \cdots + \beta_1s + \beta_0 \quad (20.3)$$

One way to define state variables is by introducing the auxiliary variable w which satisfies the differential equation:

$$q(s)w(t) = u(t) \quad (20.4)$$

The state variables can then be chosen as derivatives of w . Furthermore the output is related to this auxiliary variable as follows:

$$y(t) = p(s)w(t) \quad (20.5)$$

The proof in the next three equations shows that the introduction of this variable w does not change the system in any way. The first equation uses a simple substitution based on the differential equation (20.4). Then the order of $p(s)$ and $q(s)$ are interchanged. Lastly, y is substituted in place of $p(s)w(t)$ (using output equation (20.5)). The result is the original equation describing our system.

$$p(s)q(s)w(t) = p(s)u(t) \quad (20.6)$$

$$q(s)p(s)w(t) = p(s)u(t) \quad (20.7)$$

$$q(s)y(t) = p(s)u(t) \quad (20.8)$$

¹This content is available online at <http://cnx.org/content/m2105/2.11/>.

Using this auxillary variable, we can directly write the A , B and C matrices. A is the companion-form matrix; its last row (except for a 0 in the first position) contains the alpha coefficients from the $q(s)$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{n-1} \end{pmatrix} \quad (20.9)$$

The B vector has zeros except for the n -th row which is a 1.

$$B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad (20.10)$$

C can be expressed as

$$C = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \end{pmatrix} \quad (20.11)$$

When all of these conditions are met, the state is

$$x = \begin{pmatrix} w \\ sw \\ s^2w \\ \vdots \\ s^{n-1}w \end{pmatrix} \quad (20.12)$$

In conclusion, if the degree of p is less than that of q , we can obtain a state-space representation by inserting the coefficients of p and q in the matrices A , B and C as shown above.

Chapter 21

Partial Fraction Expansion¹

Splitting up a ratio of large polynomials into a sum of ratios of small polynomials can be a useful tool, especially for many problems involving Laplace-like transforms. This technique is known as partial fraction expansion. Here's an example of one ratio being split into a sum of three simpler ratios:

$$\frac{8x^2 + 3x - 21}{x^3 - 7x - 6} = \frac{1}{x + 2} + \frac{3}{x - 3} + \frac{4}{x + 1} \quad (21.1)$$

There are several methods for expanding a rational function via partial fractions. These include the method of clearing fractions, the Heaviside "cover-up" method, and different combinations of these two. For many cases, the Heaviside "cover-up" method is the easiest, and is therefore the method that we will introduce here. For a more complete discussion, see *Signal Processing and Linear Systems* by B.P. Lathi, Berkeley-Cambridge Press, 1998, pp-24-33. Some of the material below is based upon this book.

21.1 Heaviside "Cover-Up" Method

21.1.1 No Repeated Roots

Let's say we have a proper function $G(x) = \frac{N(x)}{D(x)}$ (by proper we mean that the degree m of the numerator $N(x)$ is less than the degree p of denominator $D(x)$). In this section we assume that there are no repeated roots of the polynomial $D(x)$.

The first step is to factor the denominator $D(x)$:

$$G(x) = \frac{N(x)}{(x - a_1)(x - a_2)\dots(x - a_p)} \quad (21.2)$$

where $a_1 \dots a_p$ are the roots of $D(x)$. We can then rewrite $G(x)$ as a sum of partial fractions:

$$G(x) = \frac{\alpha_1}{x - a_1} + \frac{\alpha_2}{x - a_2} + \dots + \frac{\alpha_p}{x - a_p} \quad (21.3)$$

where $a_1 \dots a_p$ are constants. Now, to complete the process, we must determine the values of these α coefficients. Let's look at how to find α_1 . If we multiply both sides of the equation of $G(x)$ as a sum of partial fractions (21.3) by $x - a_1$ and then let $x = a_1$, all of the terms on the right-hand side will go to zero except for α_1 . Therefore, we'll be left over with:

$$\alpha_1 = (x - a_1) G(x) \Big|_{x=a_1} \quad (21.4)$$

¹This content is available online at <<http://cnx.org/content/m2111/2.14/>>.

We can easily generalize this to a solution for any one of the unknown coefficients:

$$\alpha_r = (x - a_r) G(x) \Big|_{x=a_r} \quad (21.5)$$

This method is called the "cover-up" method because multiplying both sides by $x - a_r$ can be thought of as simply using one's finger to cover up this term in the denominator of $G(x)$. With a finger over the term that would be canceled by the multiplication, you can plug in the value $x = a_r$ and find the solution for α_r .

Example 21.1

In this example, we'll work through the partial fraction expansion of the ratio of polynomials introduced above. Before doing a partial fraction expansion, you must make sure that the ratio you are expanding is proper. If it is not, you should do long division to turn it into the sum of a proper fraction and a polynomial. Once this is done, the first step is to factor the denominator of the function:

$$\frac{8x^2 + 3x - 21}{x^3 - 7x - 6} = \frac{8x^2 + 3x - 21}{(x + 2)(x - 3)(x + 1)} \quad (21.6)$$

Now, we set this factored function equal to a sum of smaller fractions, each of which has one of the factored terms for a denominator.

$$\frac{8x^2 + 3x - 21}{(x + 2)(x - 3)(x + 1)} = \frac{\alpha_1}{x + 2} + \frac{\alpha_2}{x - 3} + \frac{\alpha_3}{x + 1} \quad (21.7)$$

To find the alpha terms, we just cover up the corresponding denominator terms in $G(x)$ and plug in the root associated with the alpha:

$$\begin{aligned} \alpha_1 &= (x + 2) G(x) \Big|_{x=-2} \\ &= \frac{8x^2 + 3x - 21}{(x - 3)(x + 1)} \Big|_{x=-2} \\ &= 1 \end{aligned} \quad (21.8)$$

$$\begin{aligned} \alpha_2 &= (x - 3) G(x) \Big|_{x=3} \\ &= \frac{8x^2 + 3x - 21}{(x + 2)(x + 1)} \Big|_{x=3} \\ &= 3 \end{aligned} \quad (21.9)$$

$$\begin{aligned} \alpha_3 &= (x + 3) G(x) \Big|_{x=-1} \\ &= \frac{8x^2 + 3x - 21}{(x + 2)(x - 3)} \Big|_{x=-1} \\ &= 4 \end{aligned} \quad (21.10)$$

We now have our completed partial fraction expansion:

$$\frac{8x^2 + 3x - 21}{(x + 2)(x - 3)(x + 1)} = \frac{1}{x + 2} + \frac{3}{x - 3} + \frac{4}{x + 1} \quad (21.11)$$

21.1.2 Repeated Roots

When the function $G(x)$ has a repeated root in its denominator, as in

$$G(x) = \frac{N(x)}{(x-b)^r(x-a_1)(x-a_2)\dots(x-a_j)} \quad (21.12)$$

Somewhat more special care must be taken to find the partial fraction expansion. The non-repeated terms are expanded as before, but for the repeated root, an extra fraction is added for each instance of the repeated root:

$$G(x) = \frac{\beta_0}{(x-b)^r} + \frac{\beta_1}{(x-b)^{r-1}} + \dots + \frac{\beta_{r-1}}{x-b} + \frac{\alpha_1}{x-a_1} + \frac{\alpha_2}{x-a_2} + \dots + \frac{\alpha_j}{x-a_j} \quad (21.13)$$

All of the alpha constants can be found using the non-repeated roots method above. Finding the beta coefficients (which are due to the repeated root) has the same Heaviside feel to it, except that this time we will add a twist by using the derivative to eliminate some unwanted terms.

Starting off directly with the cover-up method, we can find β_0 . By multiplying both sides by $(x-b)^r$, we'll get:

$$(x-b)^r G(x) = \beta_0 + \beta_1(x-b) + \dots + \beta_{r-1}(x-b)^{r-1} + \alpha_1 \frac{(x-b)^r}{x-a_1} + \alpha_2 \frac{(x-b)^r}{x-a_2} + \dots + \alpha_j \frac{(x-b)^r}{x-a_j} \quad (21.14)$$

Now that we have "covered up" the $(x-b)^r$ term in the denominator of $G(x)$, we plug in $x=b$ to each side; this cancels every term on the right-hand side except for β_0 , leaving the formula

$$\beta_0 = (x-b)^r G(x) \Big|_{x=b} \quad (21.15)$$

To find the other values of the beta coefficients, we can take advantage of the derivative. By taking the derivative of the equation after cover-up (21.14) (with respect to x the right-hand side becomes β_1 plus terms containing an $x-b$ in the numerator. Again, plugging in $x=b$ eliminates everything on the right-hand side except for β_1 , leaving us with a formula for β_1 :

$$\beta_1 = \frac{d(x-b)^r G(x)}{dx} \Big|_{x=b} \quad (21.16)$$

Generalizing over this pattern, we can continue to take derivatives to find the other beta terms. The solution for all beta terms is

$$\beta_k = \frac{1}{k!} \frac{d^k(x-b)^r G(x)}{dx^k} \Big|_{x=b} \quad (21.17)$$

NOTE: To check if you've done the partial fraction expansion correctly, just add all of the partial fractions together to see if their sum equals the original ratio of polynomials.

21.2 Finding Partial Fractions in Matlab

Matlab can be a useful tool in finding partial fraction expansions when the ratios become too unwieldy to expand by hand. It can handle symbolic variables. For example, if you type `syms s`, s will be treated as a symbolic variable. You can then use it as such when you make function assignments.

If you've done this and have then made a function, say $H(s)$, which is a ratio of two polynomials in the symbolic variable s , there are two ways to get the partial fraction expansion of it. A trick way is to say `diff(int(H))`. When you use these functions together, Matlab gives back H expanded into partial fractions. There's also a more formal way to do it using the `residue` command. Type `help residue` in Matlab for details.

Chapter 22

Linear Time Invariant Systems¹

22.1 Introduction

Linearity and time invariance are two system properties that greatly simplify the study of systems that exhibit them. In our study of signals and systems, we will be especially interested in systems that demonstrate both of these properties, which together allow the use of some of the most powerful tools of signal processing.

22.2 Linear Time Invariant Systems

22.2.1 Linear Systems

If a system is linear, this means that when an input to a given system is scaled by a value, the output of the system is scaled by the same amount.

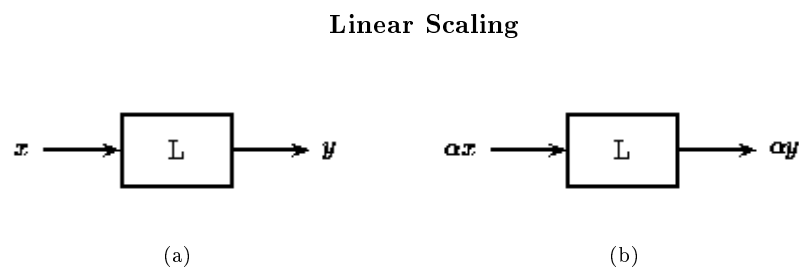


Figure 22.1

In Figure 22.1(a) above, an input x to the linear system L gives the output y . If x is scaled by a value α and passed through this same system, as in Figure 22.1(b), the output will also be scaled by α .

A linear system also obeys the principle of superposition. This means that if two inputs are added together and passed through a linear system, the output will be the sum of the individual inputs' outputs.

¹This content is available online at <http://cnx.org/content/m2102/2.24/>.

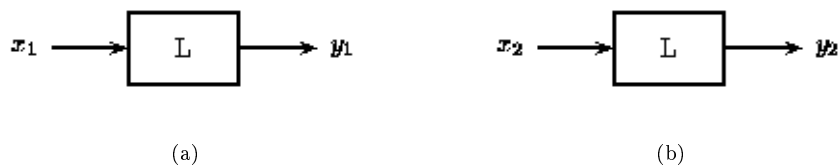


Figure 22.2

Superposition Principle

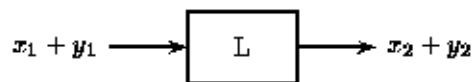


Figure 22.3: If Figure 22.2 is true, then the principle of superposition says that Figure 22.3 (Superposition Principle) is true as well. This holds for linear systems.

That is, if Figure 22.2 is true, then Figure 22.3 (Superposition Principle) is also true for a linear system. The scaling property mentioned above still holds in conjunction with the superposition principle. Therefore, if the inputs x and y are scaled by factors α and β , respectively, then the sum of these scaled inputs will give the sum of the individual scaled outputs:

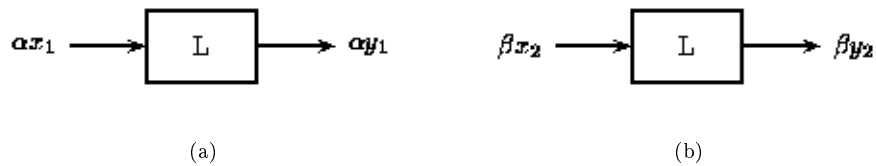


Figure 22.4

Superposition Principle with Linear Scaling

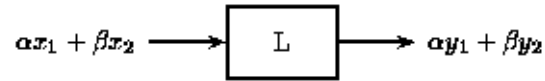


Figure 22.5: Given Figure 22.4 for a linear system, Figure 22.5 (Superposition Principle with Linear Scaling) holds as well.

Example 22.1

Consider the system H_1 in which

$$H_1(f(t)) = tf(t) \quad (22.1)$$

for all signals f . Given any two signals f, g and scalars a, b

$$H_1(af(t) + bg(t)) = t(af(t) + bg(t)) = atf(t) + btg(t) = aH_1(f(t)) + bH_1(g(t)) \quad (22.2)$$

for all real t . Thus, H_1 is a linear system.

Example 22.2

Consider the system H_2 in which

$$H_2(f(t)) = (f(t))^2 \quad (22.3)$$

for all signals f . Because

$$H_2(2t) = 4t^2 \neq 2t^2 = 2H_2(t) \quad (22.4)$$

for nonzero t , H_2 is not a linear system.

22.2.2 Time Invariant Systems

A time-invariant system has the property that a certain input will always give the same output (up to timing), without regard to when the input was applied to the system.

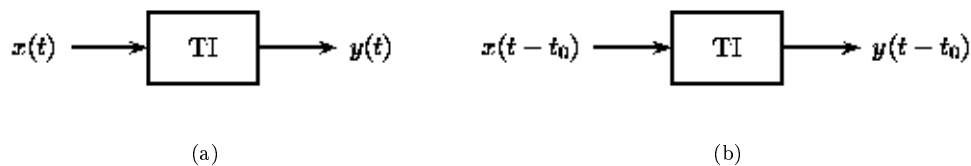
Time-Invariant Systems


Figure 22.6: Figure 22.6(a) shows an input at time t while Figure 22.6(b) shows the same input t_0 seconds later. In a time-invariant system both outputs would be identical except that the one in Figure 22.6(b) would be delayed by t_0 .

In this figure, $x(t)$ and $x(t - t_0)$ are passed through the system TI. Because the system TI is time-invariant, the inputs $x(t)$ and $x(t - t_0)$ produce the same output. The only difference is that the output due to $x(t - t_0)$ is shifted by a time t_0 .

Whether a system is time-invariant or time-varying can be seen in the differential equation (or difference equation) describing it. **Time-invariant systems are modeled with constant coefficient equations.** A constant coefficient differential (or difference) equation means that the parameters of the system are **not** changing over time and an input now will give the same result as the same input later.

Example 22.3

Consider the system H_1 in which

$$H_1(f(t)) = tf(t) \quad (22.5)$$

for all signals f . Because

$$S_T(H_1(f(t))) = S_T(tf(t)) = (t - T)f(t - T) \neq tf(t - T) = H_1(f(t - T)) = H_1(S_T(f(t))) \quad (22.6)$$

for nonzero T , H_1 is not a time invariant system.

Example 22.4

Consider the system H_2 in which

$$H_2(f(t)) = (f(t))^2 \quad (22.7)$$

for all signals f . For all real T and signals f ,

$$S_T(H_2(f(t))) = S_T(f(t)^2) = (f(t - T))^2 = H_2(f(t - T)) = H_2(S_T(f(t))) \quad (22.8)$$

for all real t . Thus, H_2 is a time invariant system.

22.2.3 Linear Time Invariant Systems

Certain systems are both linear and time-invariant, and are thus referred to as LTI systems.

Linear Time-Invariant Systems

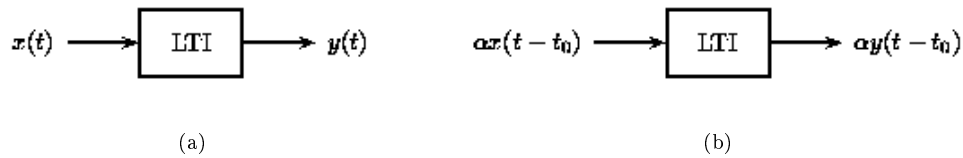


Figure 22.7: This is a combination of the two cases above. Since the input to Figure 22.7(b) is a scaled, time-shifted version of the input in Figure 22.7(a), so is the output.

As LTI systems are a subset of linear systems, they obey the principle of superposition. In the figure below, we see the effect of applying time-invariance to the superposition definition in the linear systems section above.

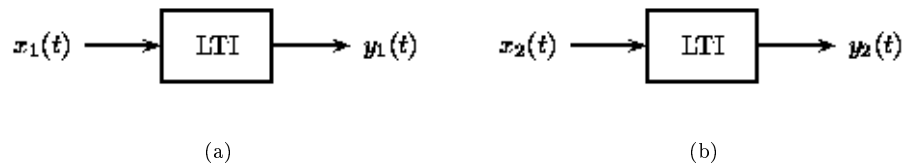


Figure 22.8

Superposition in Linear Time-Invariant Systems

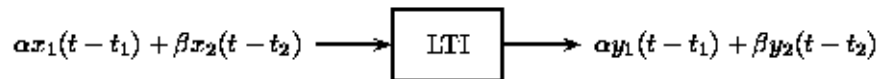
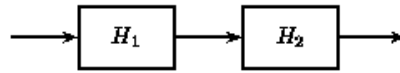


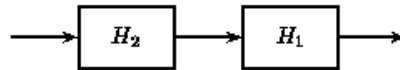
Figure 22.9: The principle of superposition applied to LTI systems

22.2.3.1 LTI Systems in Series

If two or more LTI systems are in series with each other, their order can be interchanged without affecting the overall output of the system. Systems in series are also called cascaded systems.

Cascaded LTI Systems


(a)

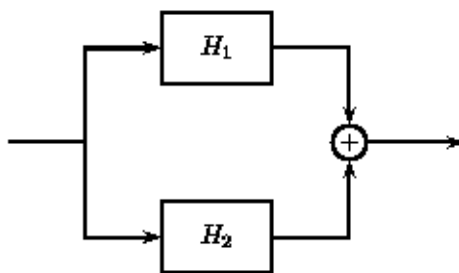


(b)

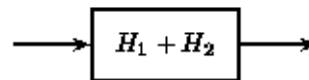
Figure 22.10: The order of cascaded LTI systems can be interchanged without changing the overall effect.

22.2.3.2 LTI Systems in Parallel

If two or more LTI systems are in parallel with one another, an equivalent system is one that is defined as the sum of these individual systems.

Parallel LTI Systems


(a)



(b)

Figure 22.11: Parallel systems can be condensed into the sum of systems.

Example 22.5

Consider the system H_3 in which

$$H_3(f(t)) = 2f(t) \quad (22.9)$$

for all signals f . Given any two signals f, g and scalars a, b

$$H_3(af(t) + bg(t)) = 2(af(t) + bg(t)) = a2f(t) + b2g(t) = aH_3(f(t)) + bH_3(g(t)) \quad (22.10)$$

for all real t . Thus, H_3 is a linear system. For all real T and signals f ,

$$S_T(H_3(f(t))) = S_T(2f(t)) = 2f(t - T) = H_3(f(t - T)) = H_3(S_T(f(t))) \quad (22.11)$$

for all real t . Thus, H_3 is a time invariant system. Therefore, H_3 is a linear time invariant system.

Example 22.6

As has been previously shown, each of the following systems are not linear or not time invariant.

$$H_1(f(t)) = tf(t) \quad (22.12)$$

$$H_2(f(t)) = (f(t))^2 \quad (22.13)$$

Thus, they are not linear time invariant systems.

22.3 Linear Time Invariant Demonstration

Image not finished

Figure 22.12: Interact(when online) with the Mathematica CDF above demonstrating Linear Time Invariant systems. To download, right click and save file as .cdf.

22.4 LTI Systems Summary

Two very important and useful properties of systems have just been described in detail. The first of these, linearity, allows us the knowledge that a sum of input signals produces an output signal that is the summed original output signals and that a scaled input signal produces an output signal scaled from the original output signal. The second of these, time invariance, ensures that time shifts commute with application of the system. In other words, the output signal for a time shifted input is the same as the output signal for the original input signal, except for an identical shift in time. Systems that demonstrate both linearity and time invariance, which are given the acronym LTI systems, are particularly simple to study as these properties allow us to leverage some of the most powerful tools in signal processing.

Chapter 23

I/O and I/S/O Relationships in Time and Frequency¹

23.1 I/O and I/S/O representation of SISO linear systems

I/O	I/S/O
variables: (u, y)	variables: (u, x, y)
$\frac{dq}{dt}y(t)$ $\frac{dp}{dt}u(t), n =$ $\deg(q) \geq$ $\deg(p)$	$\frac{d}{dt}x(t) = Ax(t) + Bu(t), y(t) = Cx(t) + Du(t)$
$u(t), y(t) \in \mathbb{R}$	$u(t) \in \mathbb{R}^n, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}$

Impulse Response

$\frac{dq}{dt}h(t) = \frac{dp}{dt}\delta(t)$	$h(t) = D\delta(t) + Ce^{At}B, t \geq 0$
$H(s) = \mathcal{L}[h(t)] = \frac{p(s)}{q(s)}$	$H(s) = D + C(sI - A)^{-1}B$

Poles - characteristic roots - eigenfrequencies

$\lambda_i, q(\lambda_i) = 0, I = 1, \dots, n$	$\det(\lambda_i I - A) = 0$
--	-----------------------------

Zeros

$H(z_i) = 0 \Leftrightarrow p(z_i), 1, \dots, n$	$\det \begin{pmatrix} z_i I - A & -B \\ -C & -D \end{pmatrix} = 0$
--	--

Matrix exponential

$\left(e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) \Rightarrow \left(\frac{d}{dt} (e^{At}) = Ae^{At} = e^{At}A \right)$
$\mathcal{L}[e^{At}] = (sI - A)^{-1}$

¹This content is available online at <<http://cnx.org/content/m10511/2.5/>>.

BIBO stability
$y = (h, u)$, requirement
$\exists \forall u : u : ((\ u\ _\infty < \infty) \Rightarrow ((u)_\infty < \infty))$
$\Leftrightarrow (h)_1 = \int_0^\infty h(t) dt < \infty$
$\Leftrightarrow \Re(\lambda_i) < 0 \Leftrightarrow \text{poles} \in \text{LHP}$

Solution in the time domain	
$y(t) = y_{zi}(t) + y_{zs}(t)$	$x(t) = x_{zi}(t) + x_{zs}(t)$
$y(t) = \sum_{i=1}^n c_i e^{\lambda_i t} + \int_{0^-}^t h(t-\tau) u(\tau) d\tau$	$x(t) = e^{At} x(0^-) + \int_{0^-}^t e^{A(t-\tau)} B u(\tau) d\tau$
	$y(t) = C e^{At} x(0^-) + \int_{0^-}^t (D\delta(t-\tau) + C e^{A(t-\tau)} B) u(\tau) d\tau, h(\cdot) = D\delta(t-\tau) + C e^{A(t-\tau)} B$
	$y(t) = C e^{At} x(0^-) + \int_{0^-}^t h(t-\tau) u(\tau) d\tau$
Laplace Transform: Solution in the frequency domain	
$Y(s) = \frac{r(s)}{q(s)} + H(s) U(s)$	$X(s) = (sI - A)^{-1} x(0^-) + (sI - A)^{-1} B U(s)$
	$Y(s) = C(sI - A)^{-1} x(0^-) + (D + C(sI - A)^{-1} B) U(s), H(s) = D + C(sI - A)^{-1} B$

Table 23.1

23.2 Definition of state from I/O description

Let $H(s) = D + \frac{\bar{p}(s)}{q(s)}$, $\deg(\bar{p}) < \deg(q)$. Define w so that $\frac{dq}{dt} w(t) = u(t)$, $\left(y(t) = \frac{d\bar{p}}{dt} w + Du(t) \right) \Rightarrow \left(x^T = \begin{pmatrix} w & w^1 & \dots & w^{n-1} \end{pmatrix} \in \mathbb{R}^n \right)$, n : degree of $q(s)$.

23.3 Various Responses

Definition 23.1: Zero-input or free response

response due to initial conditions alone.

Definition 23.2: Zero-state or forced response

response due to input (forcing function) alone (zero initial condition).

Definition 23.3: Homogeneous solution

general form of free-response (arbitrary initial conditions).

Definition 23.4: Particular solution

forced response.

Definition 23.5: Steady-state response

response obtained for large values of time $T \rightarrow \infty$.

Definition 23.6: Transient response

full response minus steady-state response.

Glossary

A Algebraic Multiplicity

The number of repetitions of a certain eigenvalue. If, for a certain matrix, $\lambda = \{3, 3, 4\}$, then the algebraic multiplicity of 3 would be 2 (as it appears twice) and the algebraic multiplicity of 4 would be 1 (as it appears once). This type of multiplicity is normally represented by the Greek letter α , where $\alpha(\lambda_i)$ represents the algebraic multiplicity of λ_i .

G Geometric Multiplicity

A particular eigenvalue's geometric multiplicity is defined as the dimension of the nullspace of $\lambda I - A$. This type of multiplicity is normally represented by the Greek letter γ , where $\gamma(\lambda_i)$ represents the geometric multiplicity of λ_i .

H Homogeneous solution

general form of free-response (arbitrary initial conditions).

P Particular solution

forced response.

S Steady-state response

response obtained for large values of time $T \rightarrow \infty$.

T Transient response

full response minus steady minus state response.

Z Zero-input or free response

response due to initial conditions alone.

Zero-state or forced response

response due to input (forcing function) alone (zero initial condition).

Index of Keywords and Terms

Keywords are listed by the section with that keyword (page numbers are in parentheses). Keywords do not necessarily appear in the text of the page. They are merely associated with that section. *Ex.* apples, § 1.1 (1) **Terms** are referenced by the page they appear on. *Ex.* apples, 1

- A** adjoint, § 18(47)
Algebraic Multiplicity, 33
- B** bases, § 13(35)
basis, § 13(35), 35, 35
- C** cascade, § 22(61)
cayley, § 11(29)
characteristic, § 11(29)
completely observable, 14
constant coefficient, § 9(21)
controllability, 5, § 5(11)
convolution, § 17(45)
cover-up, § 21(57)
- D** decomposition, § 12(31)
determinant, § 2(3), § 18(47)
diagonal, § 7(15)
diff eq, § 20(55)
differential equation, § 20(55)
dynamical, § 10(25)
- E** eigenvalue, § 7(15), § 12(31)
eigenvector, § 7(15), § 12(31)
- F** forced, § 8(17)
fraction, § 21(57)
- G** generating set, § 13(35), 35
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grammian, § 5(11)
- H** hamilton, § 11(29)
heaviside, § 21(57)
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Homogeneous solution, 70
- I** introduction to state space, § 1(1)
inverse, § 18(47)
- L** laplace, § 8(17), § 15(41), § 16(43)
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- M** matrix, § 14(37), § 18(47), § 19(49)
matrix exponential, § 14(37), 37
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memory, § 4(9), § 10(25)
multiplicity, § 12(31)
- N** n-th order, § 20(55)
non-zero initial condition, § 9(21)
- O** observability, § 5(11), § 6(13)
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- P** parallel, § 22(61)
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- R** repeated root, § 21(57)
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- S** solution, § 16(43)
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- V** vector space, § 13(35)
- Z** Zero-input or free response, 70
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