Gravitation fundamentals

By:

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C O N N E X I O N S

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Chapter 1

Gravitation[1]

Gravitation is an inherent property of all matter. Two bodies attract each other by virtue of their mass. This force between two bodies of any size (an atom or a galaxy) signifies existence of matter and is known as gravitational force.

Gravitational force is weakest of four fundamental forces. It is, therefore, experienced only when at least one of two bodies has considerable mass. This presents difficulties in setting up illustrations with terrestrial objects. On the other hand, gravitation is the force that sets up our universe and governs motions of all celestial bodies. Orbital motions of satellites – both natural and artificial – are governed by gravitational force.

Newton derived a law to quantify gravitational force between two "particles". The famous incidence of an apple falling from a tree stimulated Newton's mind to analyze observations and carry out series of calculations that finally led him to propose universal law of gravitation. A possible sequence of reasoning, leading to the postulation is given here :

1: The same force of attraction works between "Earth (E) and an apple (A)" and between "Earth (E) and Moon (M)".

2: From the analysis of data available at that time, he observed that the ratio of forces of attraction for the above two pairs is equal to the ratio of square of distance involved as :

[1]This content is available online at <http://cnx.org/content/m15085/1.3/>.

Gravitational force

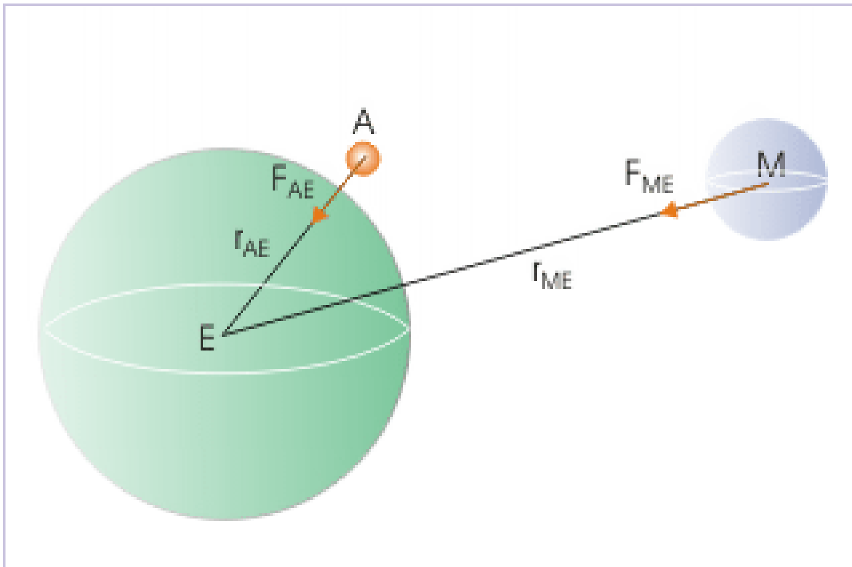

Figure 1.1: Earth - moon - apple system

$$\frac{F_{ME}}{F_{AE}} = \frac{r_{AE^2}}{r_{ME^2}}$$

3: From above relation, Newton concluded that the force of attraction between any pair of two bodies is inversely proportional to the square of linear distance between them.

$$F \propto \frac{1}{r^2}$$

4: From second law of motion, force is proportional to mass of the body being subjected to gravitational force. From third law of motion, forces exist in equal and opposite pair. Hence, gravitational force is also proportional to the mass of other body. Newton concluded that force of gravitation is proportional to the product of mass of two bodies.

$$F \propto m_1 m_2$$

5: Combining two "proportional" equations and introducing a constant of proportionality,"G", Newton proposed the gravitational law as :

$$\Rightarrow F = \frac{G m_1 m_2}{r^2}$$

In order to emphasize the universal character of gravitational force, the constant "G" is known as "Universal gravitational constant". Its value is :

$$G = 6.67X10^{-11} \quad N - m^2/kg^2$$

This law was formulated based on the observations on real bodies - small (apple) and big (Earth and moon). However, Newton's law of gravitation is stated strictly for two particles. The force pair acts between two particles, along the line joining their positions as shown in the figure :

Gravitational force

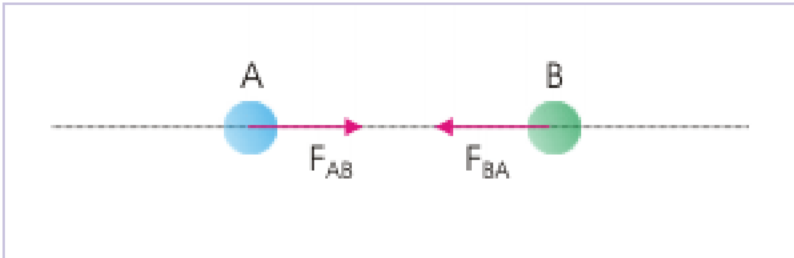

Figure 1.2: Force between two particles placed at a distance

Extension of this law to real bodies like Earth and Moon pair can be understood as bodies are separated by large distance (about $0.4X10^6$ km), compared to dimension of bodies (in thousands km). Two bodies, therefore, can be treated as particles.

On the other hand, Earth can not be treated as particle for "Earth and Apple" pair, based on the reasoning of large distance. Apple is right on the surface of Earth. Newton proved a theorem that every spherical shell (hollow sphere) behaves like a particle for a particle external to it. Extending this theorem, Earth, being continuous composition of infinite numbers of shells of different radii, behaves - as a whole - like a particle for external object like an apple.

As a matter of fact, we will prove this theorem, employing gravitational field concept for a spherical mass like that of Earth. For the time being, we consider Earth and apple as particles, based on the Newton's shell theory. In that case, the distance between Apple and center of Earth is equal to the radius of Earth i.e 6400 km.

1.1 Magnitude of force

The magnitude of gravitational force between terrestrial objects is too small to experience. A general question that arises in the mind of a beginner is "why do not we experience this force between, say, a book and pencil?" The underlying fact is that gravitational force is indeed a very small force for masses that we deal with in our immediate surrounding - except Earth.

We can appreciate this fact by calculating force of gravitation between two particle masses of 1 kg each, which are 1 m apart :

$$\Rightarrow F = \frac{6.67X10^{-11}X1X1}{1^2} = 6.67X10^{-11} \quad N$$

This is too insignificant a force to manifest against bigger forces like force of gravitation due to Earth, friction, force due to atmospheric pressure, wind etc.

Evidently, this is the small value of "G", which renders force of gravitation so small for terrestrial objects. Gravitation plays visible and significant role, where masses are significant like that of planets including our Earth, stars and such other massive aggregation, including "black holes" with extraordinary gravitational force to hold back even light. This is the reason, we experience gravitational force of Earth, but we do not experience gravitational force due to a building or any such structures on Earth.

1.2 Gravitational force vector

Newton's law of gravitation provides with expression of gravitational force between two bodies. Here, gravitational force is a vector. However, force vector is expressed in terms of quantities, which are not vectors. The linear distance between two masses, appearing in the denominator of the expression, can have either of two directions from one to another point mass.

Even if, we refer the linear distance between two particles to a reference direction, the vector appears in the denominator and is, then, squared also. In order to express gravitational force in vector form, therefore, we shall consider a unit vector in the reference direction and use the same to denote the direction of force as:

Direction of gravitational force

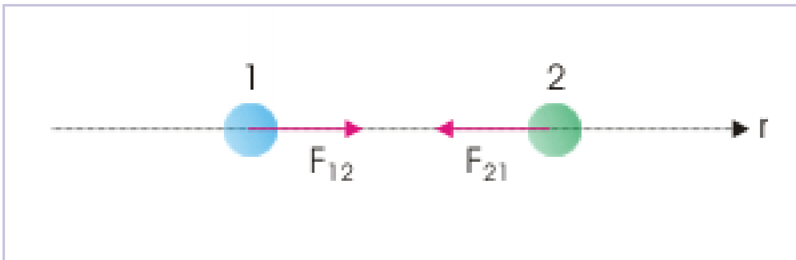

Figure 1.3: Force between two particles placed at a distance

$$\mathbf{F}_{12} = \frac{Gm_1 m_2 \ \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{F}_{21} = -\frac{Gm_1 m_2 \ \hat{\mathbf{r}}}{r^2}$$

Note that we need to put a negative sign before the second expression to make the direction consistent with the direction of gravitational force of attraction. We can easily infer that sign in the expression actually depends on the choice of reference direction.

1.3 Net gravitational force

Gravitation force is a vector quantity. The net force of gravitation on a particle is equal to resultant of forces due to all other particles. This is also known as "superposition principle", according to which net effect is sum of individual effects. Mathematically,

$$\Rightarrow \mathbf{F} = \Sigma \mathbf{F}_i$$

Here, **F** is the net force due to other particles 1, 2, 3, and so on.

An extended body is considered to be continuous aggregation of elements, which can be treated as particles. This fact can be represented by an integral of all elemental forces due to all such elements of a body, which are treated as particles. The force on a particle due to an extended body, therefore, can be computed as :

Net gravitational force

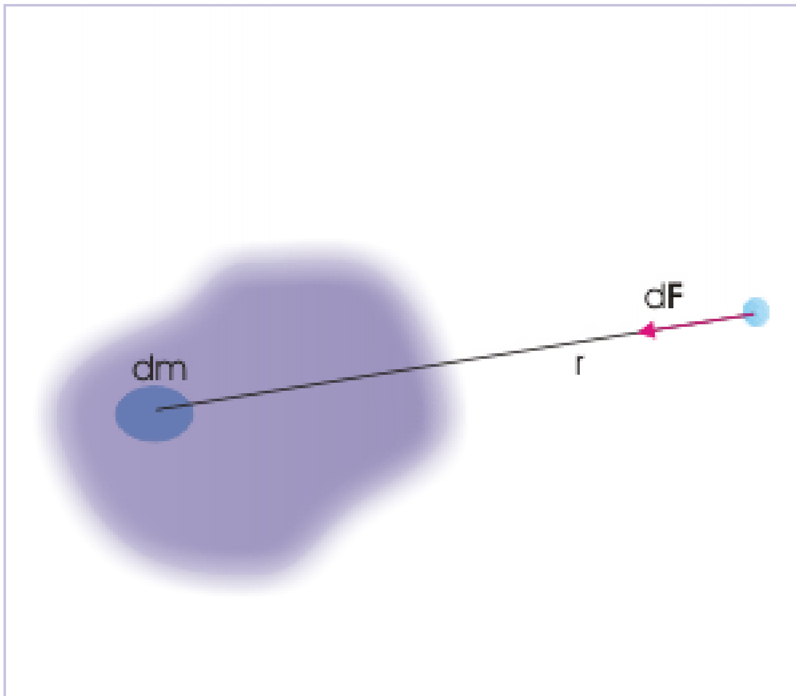

Figure 1.4: Net gravitational force is vector sum of individual gravitations due to particle like masses.

$$\mathbf{F} = \int d\mathbf{F}$$

where integration is evaluated to include all mass of a body.

1.4 Examples

1.4.1

Problem 1: Three identical spheres of mass "M" and radius "R" are assembled to be in contact with each other. Find gravitational force on any of the sphere due to remaining two spheres. Consider no external gravitational force exists.

Three identical sphered in contact

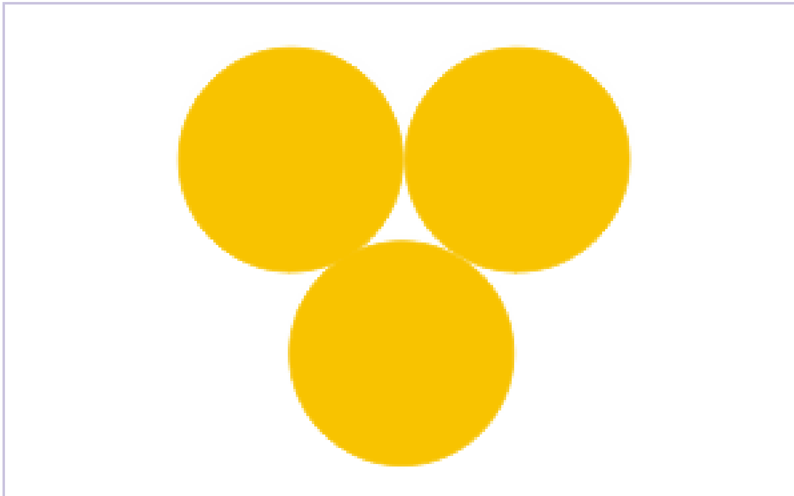

Figure 1.5: Three identical spheres of mass “M” and radius “R” are assembled in contact with each other.

Solution : The gravitational forces due to pairs of any two speres are equal in magnitude, making an angle of 60° with each other. The resultant force is :

Three identical sphered in contact

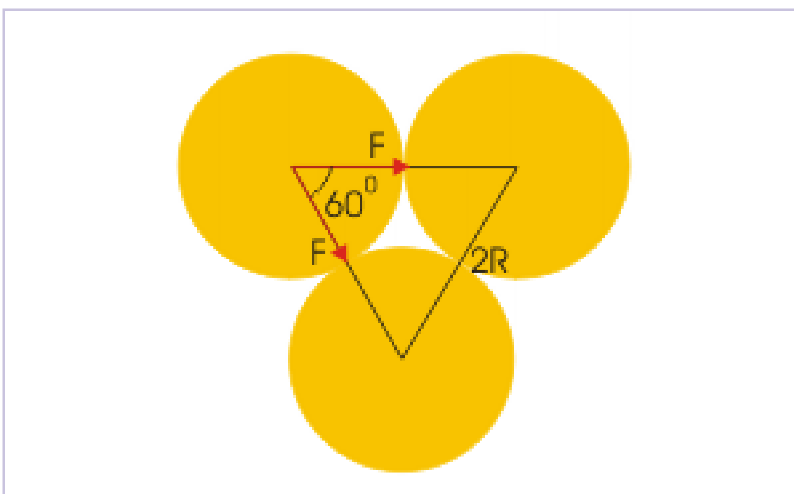

Figure 1.6: Each sphere is attracted by other two spheres.

$$\Rightarrow R = \sqrt{\left(F^2 + F^2 + 2F^2 \cos 60^0\right)}$$

$$\Rightarrow R = \sqrt{\left(2F^2 + 2F^2 X \frac{1}{2}\right)}$$

$$\Rightarrow R = \sqrt{3}F$$

Now, the distance between centers of mass of any pair of spheres is "2R". The gravitational force is :

$$F = \frac{GM^2}{(2R)^2} = \frac{GM^2}{4R^2}$$

Therefore, the resultant force on a sphere is :

$$F = \frac{\sqrt{3}GM^2}{4R^2}$$

1.4.2

Problem 2: Two identical spheres of uniform density are in contact. Show that gravitational force is proportional to the fourth power of radius of either sphere.

Solution : The gravitational force between two spheres is :

$$F = \frac{Gm^2}{(2r)^2}$$

Now, mass of each of the uniform sphere is :

$$m = \frac{4\pi r^3 X \rho}{3}$$

Putting this expression in the expression of force, we have :

$$\Rightarrow F = \frac{GX16\pi^2 r^6 \rho^2}{9X4r^2} = \frac{Gx16\pi^2 r^4 \rho^2}{36}$$

Since all other quantities are constants, including density, we conclude that gravitational force is proportional to the fourth power of radius of either sphere,

$$\Rightarrow F \propto r^4$$

1.5 Measurement of universal gravitational constant

The universal gravitational constant was first measured by Cavendish. The measurement was an important achievement in the sense that it could measure small value of "G" quite accurately.

The arrangement consists of two identical small spheres, each of mass "m". They are attached to a light rod in the form of a dumb-bell. The rod is suspended by a quartz wire of known coefficient of torsion "k" such that rod lies in horizontal plane. A mirror is attached to quartz wire, which reflects a light beam falling on it. The reflected light beam is read on a scale. The beam, mirror and scale are all arranged in one plane.

Cavendish experiment

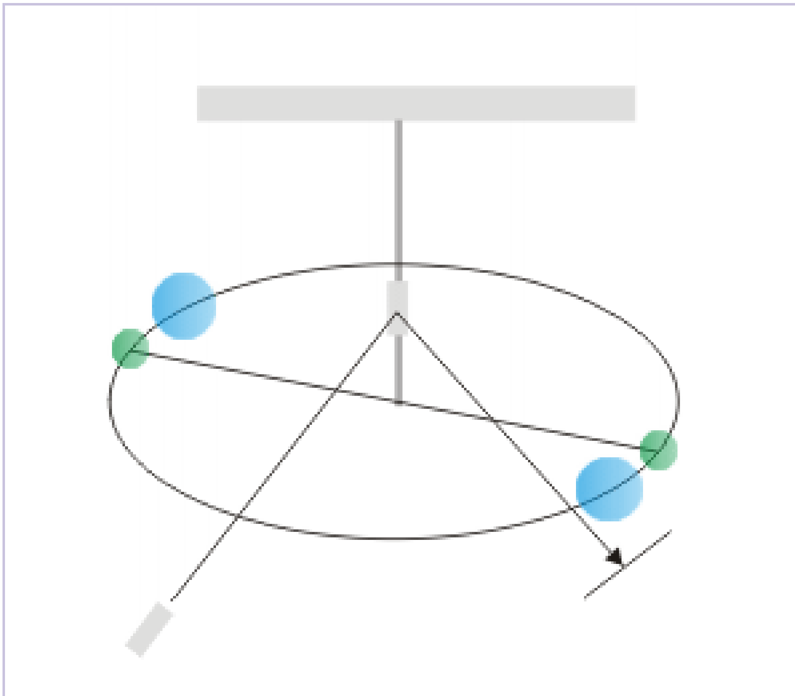

Figure 1.7: Measurement of universal gravitational constant

The rod is first made to suspend freely and stabilize at an equilibrium position. As no net force acts in the horizontal direction, the rod should rest in a position without any torsion in the quartz string. The position of the reflected light on the scale is noted. This reading corresponds to neutral position, when no horizontal force acts on the rod. The component of Earth's gravitation is vertical. Its horizontal component is zero. Therefore, it is important to keep the plane of rotation horizontal to eliminate effect of Earth's gravitation.

Two large and heavier spheres are, then brought across, close to smaller sphere such that centers of all spheres lie on a circle as shown in the figure above. The gravitational forces due to each pair of small and big mass, are perpendicular to the rod and opposite in direction. Two equal and opposite force constitutes a couple, which is given by :

$$\tau_G = F_G L$$

where "L" is the length of the rod.

The couple caused by gravitational force is balanced by the torsion in the quartz string. The torque is proportional to angle "θ" through which the rod rotates about vertical axis.

$$\tau_T = k\theta$$

The position of the reflected light is noted on the scale for the equilibrium. In this condition of equilibrium,

$$\Rightarrow F_G L = k\theta$$

Now, the expression of Newton's law of gravitation for the gravitational force is:

$$F_G = \frac{GMm}{r^2}$$

where “m” and “M” are mass of small and big spheres. Putting this in the equilibrium equation, we have :

$$\Rightarrow \frac{GMmL}{r^2} = k\theta$$

Solving for “G”, we have :

$$\Rightarrow G = \frac{r^2 k\theta}{MmL}$$

In order to improve accuracy of measurement, the bigger spheres are, then, placed on the opposite sides of the smaller spheres with respect to earlier positions (as shown in the figure below). Again, position of reflected light is noted on the scale for equilibrium position, which should lie opposite to earlier reading about the reading corresponding to neutral position.

Cavendish experiment

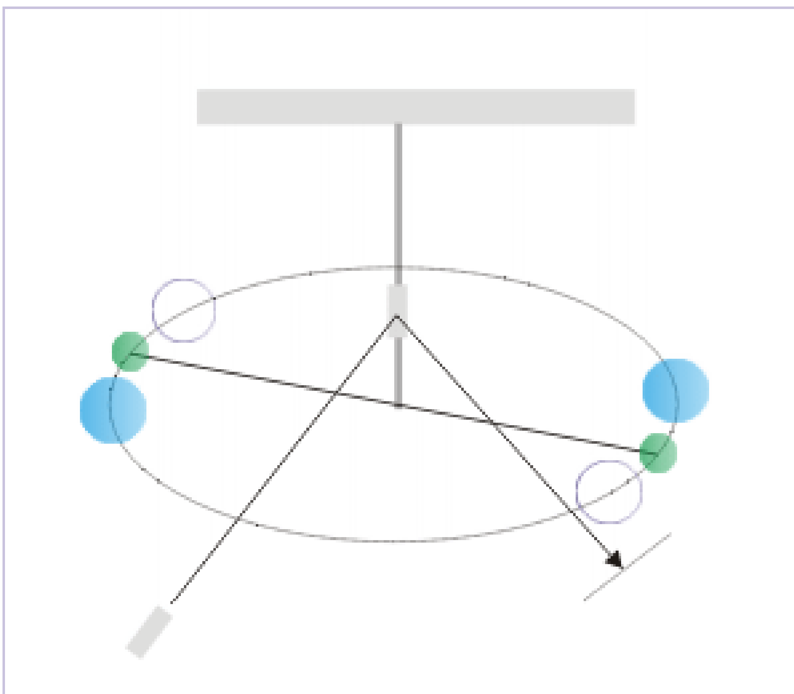

Figure 1.8: Measurement of universal gravitational constant

The difference in the readings (x) on the scale for two configurations of larger spheres is read. The distance between mirror and scale (y) is also determined. The angle subtended by the arc “x” at the mirror is twice the angle through which mirror rotates between two configurations. Hence,

Cavendish experiment

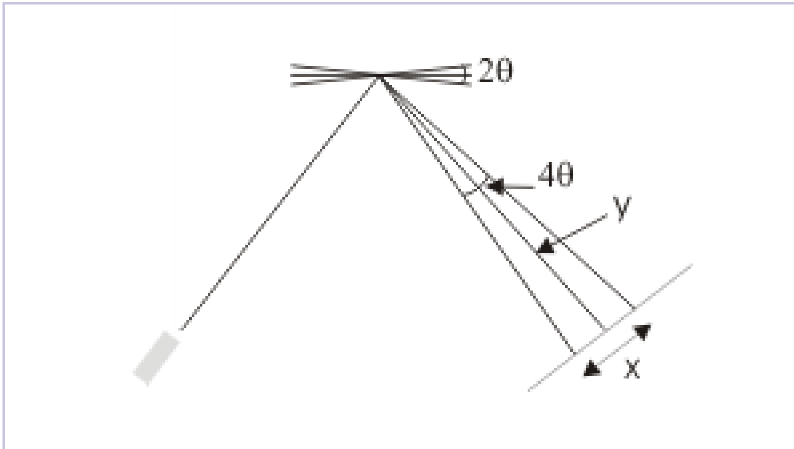

Figure 1.9: Measurement of angle

$$\Rightarrow 4\theta = \frac{x}{y}$$

$$\Rightarrow \theta = \frac{x}{4y}$$

We see here that beam, mirror and scale arrangement enables us to read an angle, which is 2 times larger than the actual angle involved. This improves accuracy of the measurement. Putting the expression of angle, we have the final expression for determination of "G",

$$G = \frac{r^2 kx}{4MmLy}$$

Chapter 2

Gravity[1]

The term "gravity" is used for the gravitation between two bodies, one of which is Earth.

Earth is composed of layers, having different densities and as such is not uniform. Its density varies from 2 kg/m^3 for crust to nearly 14 kg/m^3 for the inner core. However, inner differentiation with respect to mass is radial and not directional. This means that there is no preferential direction in which mass is aggregated more than other regions. Applying Newton's shell theorem, we can see that Earth, if considered as a solid sphere, should behave as a point mass for any point on its surface or above it.

In the nutshell, we can conclude that density difference is not relevant for a point on the surface or above it so long Earth can be considered spherical and density variation is radial and not directional. As this is approximately the case, we can treat Earth, equivalently as a sphere of uniform mass distribution, having an equivalent uniform (constant) density. Thus, force of gravitation on a particle on the surface of Earth is given by :

$$F = \frac{GMm}{R^2}$$

where "M" and "m" represents masses of Earth and particle respectively. For any consideration on Earth's surface, the linear distance between Earth and particle is constant and is equal to the radius of Earth (R).

2.1 Gravitational acceleration (acceleration due to gravity)

In accordance with Newton's second law of motion, gravity produces acceleration in the particle, which is situated on the surface. The acceleration of a particle mass "m', on the surface of Earth is obtained as :

$$\Rightarrow a = \frac{F}{m} = \frac{GM}{R^2}$$

The value corresponding to above expression constitutes the reference gravitational acceleration. However, the calculation of gravitational acceleration based on this formula would be idealized. The measured value of gravitational acceleration on the surface is different. The measured value of acceleration incorporates the effects of factors that we have overlooked in this theoretical derivation of gravitational acceleration on Earth.

We generally distinguish gravitational acceleration as calculated by above formula as " g_0 " to differentiate it from the one, which is actually measured(g) on the surface of Earth. Hence,

$$g_0 = a = \frac{F}{m} = \frac{GM}{R^2}$$

[1]This content is available online at <http://cnx.org/content/m15087/1.4/>.

This is a very significant and quite remarkable relationship. The gravitational acceleration does not dependent on the mass of the body on which force is acting! This is a special characteristic of gravitational force. For all other forces, acceleration depends on the mass of the body on which force is acting. We can easily see the reason. The mass of the body appears in both Newton's law of motion and Newton's law of gravitation. Hence, they cancel out, when two equations are equated.

2.2 Factors affecting Gravitational acceleration

The formulation for gravitational acceleration considers Earth as (i) uniform (ii) spherical and (iii) stationary body. None of these assumptions is true. As such, measured value of acceleration (g) is different to gravitational acceleration, " g_0 ", on these counts :

1. Constitution of the Earth
2. Shape of the Earth
3. Rotation of the Earth

In addition to these inherent factors resulting from the consequence of "real" Earth, the measured value of acceleration also depends on the point of measurement in vertical direction with respect to mean surface level or any reference for which gravitational acceleration is averaged. Hence, we add one more additional factor responsible for variation in the gravitational acceleration. The fourth additional factor is relative vertical position of measurement with respect to Earth's surface.

2.2.1 Constitution of the Earth

Earth is not uniform. Its density varies as we move from its center to the surface. In general, Earth can be approximated to be composed of concentric shells of different densities. For all practical purpose, we consider that the density gradation is radial and is approximated to have an equivalent uniform density within these concentric shells in all directions.

Constitution of the Earth

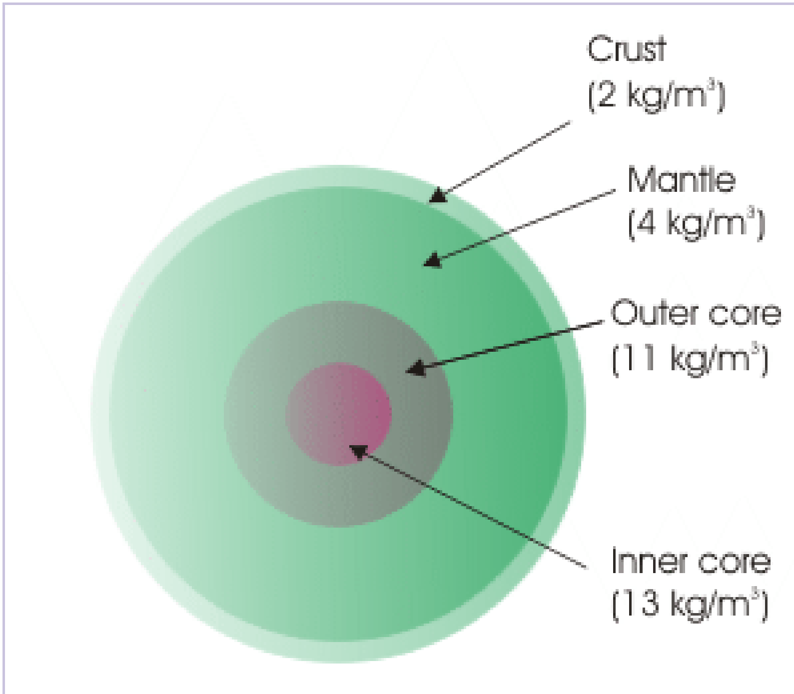

Figure 2.1: Density of Earth varies radially.

The main reason for this directional uniformity is that bulk of the material constituting Earth is fluid due to high temperature. The material, therefore, has a tendency to maintain uniform density in a given shell so conceived.

We have discussed that the radial density variation has no effect on a point on the surface or above it. This variation of density, however, impacts gravitational acceleration, when the point in the question is at a point below Earth's surface.

In order to understand the effect, let us have a look at the expression of gravitational expression :

$$g_0 = \frac{F}{m} = \frac{GM}{R^2}$$

The impact of moving down below the surface of Earth, therefore, depends on two factors

1. Mass (M) and
2. Distance from the center of Earth (r)

A point inside a deep mine shaft, for example, will result in a change in the value of gravitational acceleration due to above two factors.

We shall know subsequently that gravitational force inside a spherical shell is zero. Therefore, mass of the spherical shell above the given point does not contribute to gravitational force and hence acceleration at that point. Thus, the value of "M" in the expression of gravitational acceleration decreases as we go down from the Earth's surface. This, in turn, decreases gravitational acceleration at a point below Earth's surface. At the same time, the distance to the center of Earth decreases. This factor, in turn, increases gravitational acceleration.

If we assume uniform density, then the impact of "decrease in mass" is greater than that of impact of "decrease in distance". We shall prove this subsequently when we consider the effect of vertical position. As such, acceleration is expected to decrease as we go down from Earth's surface.

In reality the density is not uniform. Crust being relatively light and thin, the impact of first factor i.e. "decrease in mass" is less significant initially and consequently gravitational acceleration actually increases initially for some distance as we go down till it reaches a maximum value at certain point below Earth's surface. For most of depth beyond, however, gravitational acceleration decreases with depth.

2.2.2 Shape of the Earth

Earth is not a sphere. It is an ellipsoid. Its equatorial radius is greater than polar radius by 21 km. A point at pole is closer to the center of Earth. Consequently, gravitational acceleration is greater there than at the equator.

Besides, some part of Earth is protruded and some part is depressed below average level. Once again, factors of mass and distance come into picture. Again, it is the relative impact of two factors that determine the net effect. Consider a point right at the top of Mt. Everest, which is about 8.8 km from the mean sea level. Imagine incrementing radius of Earth's sphere by 8.8 km. Most of the volume so created is not filled. The proportionate increase in mass (mass of Everest mountain range) is less than that in the squared distance from the center of Earth. As such, gravitational acceleration is less than its average value on the surface. It is actually 9.80 m/s^2 as against the average of 9.81 m/s^2 , which is considered to be the accepted value for the Earth's surface.

2.2.3 Rotation of Earth

Earth rotates once about its axis of rotation in 1 day and moves around Sun in 365 days. Since Earth and a particle on Earth both move together with a constant speed around Sun, there is no effect in the measured acceleration due to gravity on the account of Earth's translational motion. The curved path around Sun can be approximated to be linear for distances under consideration. Hence, Earth can serve as inertial frame of reference for the application of Newton's law of motion, irrespective of its translational motion.

However, consideration of rotation of Earth about its axis changes the nature of Earth's reference. It is no more an inertial frame. A particle at a point, "P", is rotating about the axis of rotation. Clearly, a provision for the centripetal force should exist to meet the requirement of circular motion. We should emphasize here that centripetal force is not an additional force by itself, but is a requirement of circular motion, which should be met with by the forces operating on the particle. Here, gravitational force meets this requirement and, therefore, gets modified to that extent.

Here, we shall restrict our consideration specifically to the effect of rotation. We will ignore other factors that affect gravitational acceleration. This means that we consider Earth is a solid uniform sphere. If it is so then, measured value of acceleration is equal to reference gravitational acceleration (g_0) as modified by rotation.

As we have studied earlier, we can apply Newton's law in a non-inertial reference by providing for pseudo force. We should recall that pseudo force is applied in the direction opposite to the direction of acceleration of the frame of reference, which is centripetal acceleration in this case. The magnitude of pseudo force is equal to the product of mass of the particle and centripetal acceleration. Thus,

$$F_P = m\omega^2 r$$

After considering pseudo force, we can enumerate forces on the particle at "P" at an latitude "ϕ" as shown in the figure :

1. Pseudo force ($m\omega^2 r$)
2. Normal force (N)
3. gravitational force (mg_0)

Forces on the particle on the surface of Earth

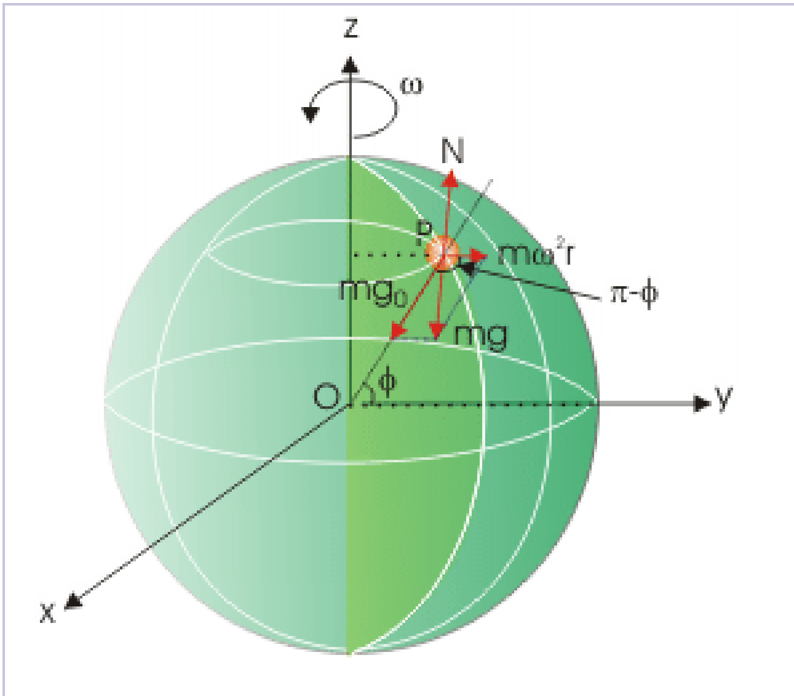

Figure 2.2: The particle is at rest under action of three forces.

The particle is subjected to normal force against the net force on the particle or the weight as measured. Two forces are equal in magnitude, but opposite in direction.

$$N = W = mg$$

It is worthwhile to note here that gravitational force and normal force are different quantities. The measured weight of the particle is equal to the product of mass and the measured acceleration. It is given by the expression "mg". On the other hand, Gravitational force is given by the Newton's equation, which considers Earth at rest. It is equal to " mg_0 ".

$$\Rightarrow F_G = \frac{GM}{R^2} = mg_0$$

Since particle is stationary on the surface of Earth, three forces as enumerated above constitute a balanced force system. Equivalently, we can say that the resultant of pseudo and gravitational forces is equal in magnitude, but opposite in direction to the normal force.

$$N = F_G + F_P$$

In other words, resultant of gravitational and pseudo forces is equal to the magnitude of measured weight of the particle. Applying parallelogram theorem for vector addition of gravitational and pseudo forces, the resultant of the two forces is :

Forces on the particle on the surface of Earth

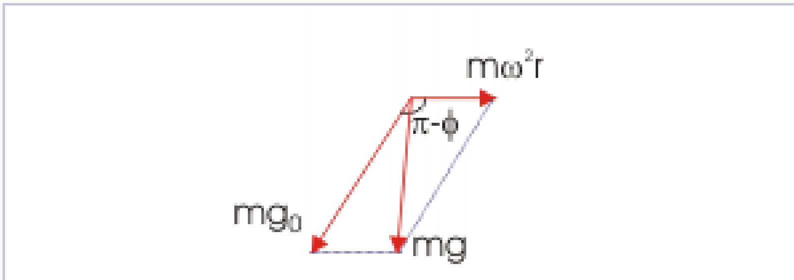

Figure 2.3: The particle is at rest under action of three forces.

$$N^2 = (mg_0)^2 + \left(m\omega^2 R\right)^2 + 2m^2\omega^2 g_0 R\cos\left(180 - \phi\right)$$

Putting N = mg and rearranging, we have:

$$\Rightarrow m^2 g^2 = m^2 g_0^2 + m^2\omega^4 R^2 - 2m^2\omega^2 g_0 R\cos\phi$$

Angular velocity of Earth is quite a small value. It may be interesting to know the value of the term having higher power of angular velocity. Since Earth completes one revolution in a day i.e an angle of "2π" in 24 hrs, the angular speed of Earth is :

$$\Rightarrow \omega = \frac{2\pi}{24X60X60} = 7.28X10^{-5} \quad \mathrm{rad}/s$$

The fourth power of angular speed is almost a zero value :

$$\Rightarrow \omega^4 = 2.8X10^{-17}$$

We can, therefore, safely neglect the term " $m^2\omega^4 r^2$ ". The expression for the measured weight of the particle, therefore, reduces to :

$$\Rightarrow g^2 = g_0^2 - 2\omega^2 g_0 R\cos\phi$$

$$\Rightarrow g = g_0\left(1 - \frac{2\omega^2 R\cos\phi}{g_0}\right)^{1/2}$$

Neglecting higher powers of angular velocity and considering only the first term of the binomial expansion,

$$\Rightarrow g = g_0\left(1 - \frac{1}{2}X\frac{2\omega^2 R\cos\phi}{g_0}\right)$$

$$\Rightarrow g = g_0 - \omega^2 R\cos\phi$$

This is the final expression that shows the effect of rotation on gravitational acceleration (mg_0). The important point here is that it is not only the magnitude that is affected by rotation, but its direction is also affected as it is no more directed towards the center of Earth.

There is no effect of rotation at pole. Being a point, there is no circular motion involved and hence, there is no reduction in the value of gravitational acceleration. It is also substantiated from the expression as latitude angle is $\phi = 90^\circ$ for the pole and corresponding cosine value is zero. Hence,

$$\Rightarrow g = g_0 - \omega^2 R\cos 90^0 = g_0$$

The reduction in gravitational acceleration is most (maximum) at the equator, where latitude angle is $\phi = 0\,^\circ$ and corresponding cosine value is maximum (=1).

$$\Rightarrow g = g_0 - \omega^2 R\cos 0^0 = g_0 - \omega^2 R$$

We can check approximate reduction at the equator, considering R = 6400 km = 6400000 m = $6.4X10^6$ m.

$$\Rightarrow \omega^2 R = \left(7.28X10^{-5}\right)^2 X \left(6.4X10^6\right) = 3.39X10^{-3} \quad m/s^2 = 0.0339 \quad m/s^2$$

This is the maximum reduction possible due to rotation. Indeed, we can neglect this variation for all practical purposes except where very high accuracy is required.

2.2.4 Vertical position

In this section, we shall discuss the effect of the vertical position of the point of measurement. For this, we shall consider Earth as a perfect sphere of radius "R" and uniform density, "ρ". Further, we shall first consider a point at a vertical height "h" from the surface and then a point at a vertical depth "d" from the surface.

2.2.4.1 Gravitational acceleration at a height

Gravitational acceleration due to Earth on its surface is equal to gravitational force per unit mass and is given by :

$$g_0 = \frac{F}{m} = \frac{GM}{R^2}$$

Gravity at an altitude

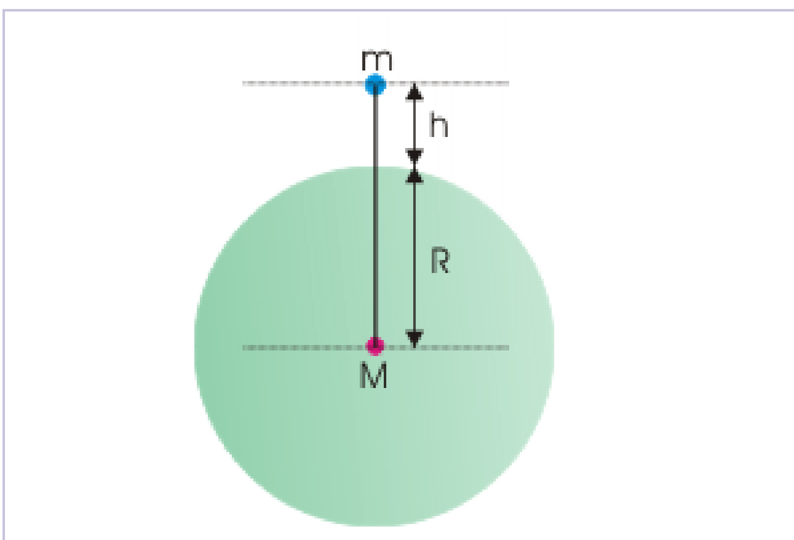

Figure 2.4: Distance between center of Earth and particle changes at an altitude.

where "M" and "R" are the mass and radius of Earth. It is clear that gravitational acceleration will decrease if measured at a height "h" from the Earth's surface. The mass of Earth remains constant, but the linear distance between particle and the center of Earth increases. The net result is that gravitational acceleration decreases to a value "g"' as given by the equation,

$$\Rightarrow g\prime = \frac{F\prime}{m} = \frac{GM}{(R+h)^2}$$

We can simplify this equation as,

$$\Rightarrow g\prime = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2}$$

Substituting for the gravitational acceleration at the surface, we have :

$$\Rightarrow g\prime = \frac{g_0}{\left(1+\frac{h}{R}\right)^2}$$

This relation represents the effect of height on gravitational acceleration. We can approximate the expression for situation where h$\ll$ R.

$$\Rightarrow g\prime = \frac{g_0}{\left(1+\frac{h}{R}\right)^2} = g_0\left(1+\frac{h}{R}\right)^{-2}$$

As h$\ll$R, we can neglect higher powers of "h/R" in the binomial expansion of the power term,

$$\Rightarrow g\prime = g_0\left(1-\frac{2h}{R}\right)$$

We should always keep in mind that this simplified expression holds for the condition, h $\ll$ R. For small vertical altitude, gravitational acceleration decreases linearly with a slope of "-2/R". If the altitude is large as in the case of a communication satellite, then we should resort to the original expression,

$$\Rightarrow g\prime = \frac{GM}{(R+h)^2}$$

If we plot gravitational acceleration .vs. altitude, the plot will be about linear for some distance.

Acceleration .vs. linear distance

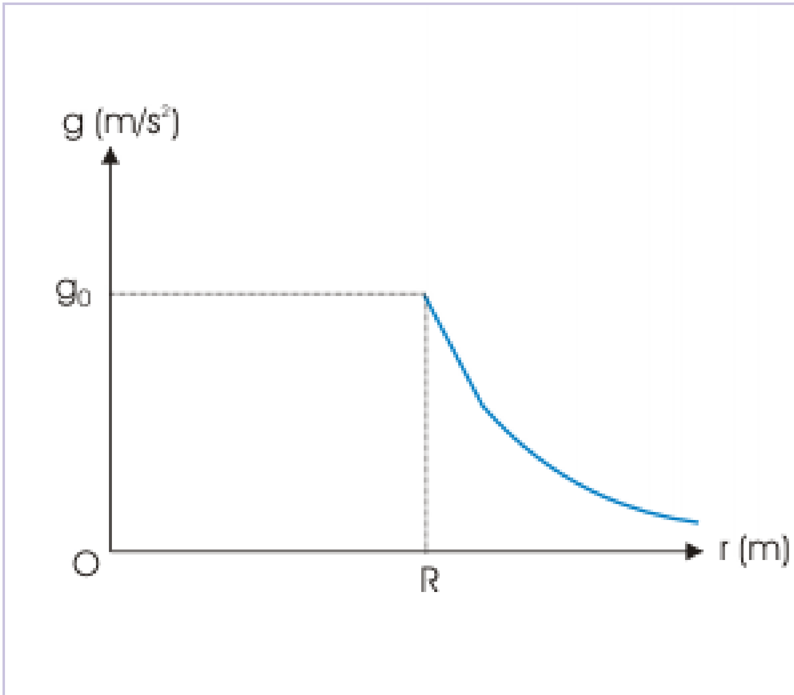

Figure 2.5: The plot shows variations in gravitational acceleration as we move vertically upwards from center of Earth.

2.2.4.2 Gravitational acceleration at a depth

In order to calculate gravitational acceleration at a depth "d", we consider a concentric sphere of radius "R-d" as shown in the figure. Here, we shall make use of the fact that gravitational force inside a spherical shell is zero. It means that gravitational force due to the spherical shell above the point is zero. On the other hand, gravitational force due to smaller sphere can be calculated by treating it as point mass. As such, net gravitational acceleration at point "P" is :

Gravity at a depth

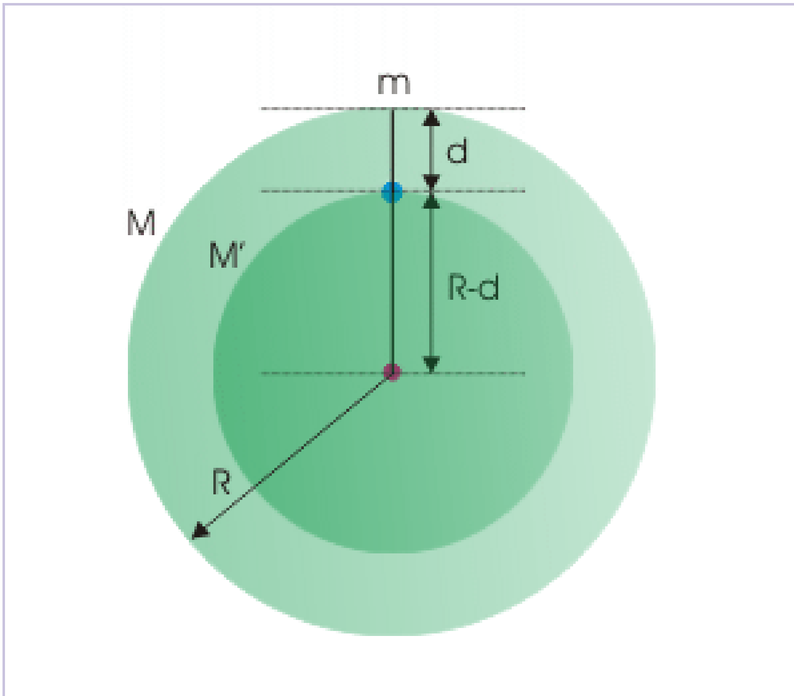

Figure 2.6: Distance between center of Earth and particle changes at an altitude.

$$\Rightarrow g\prime = \frac{F\prime}{m} = \frac{GM\prime}{(R-d)^2}$$

where "M"' is the mass of the smaller sphere. If we consider Earth as a sphere of uniform density, then :

$$M = V\rho$$

$$\Rightarrow \rho = \frac{M}{\frac{4}{3}\pi R^3}$$

Hence, mass of smaller sphere is equal to the product :

$$M\prime = \rho V\prime$$

$$\Rightarrow M\prime = \frac{\frac{4}{3}\pi (R-d)^3 XM}{\frac{4}{3}\pi R^3} = \frac{(R-d)^3 XM}{R^3}$$

Substituting in the expression of gravitational acceleration, we have :

$$\Rightarrow g\prime = \frac{G(R-d)^3 XM}{(R-d)^2 R^3}$$

Inserting gravitational acceleration at the surface ($g_0 = GM/R^2$), we have :

$$\Rightarrow g\prime = \frac{g_0 (R-d)^3}{(R-d)^2 R} = \frac{g_0 \, (R-d)}{R}$$

$$g\prime = g_0 \left(1 - \frac{d}{R}\right)$$

This is also a linear equation. We should note that this expression, unlike earlier case of a point above the surface, makes no approximation . The gravitational acceleration decreases linearly with distance as we go down towards the center of Earth. Conversely, the gravitational acceleration increases linearly with distance as we move from the center of Earth towards the surface.

Acceleration .vs. linear distance

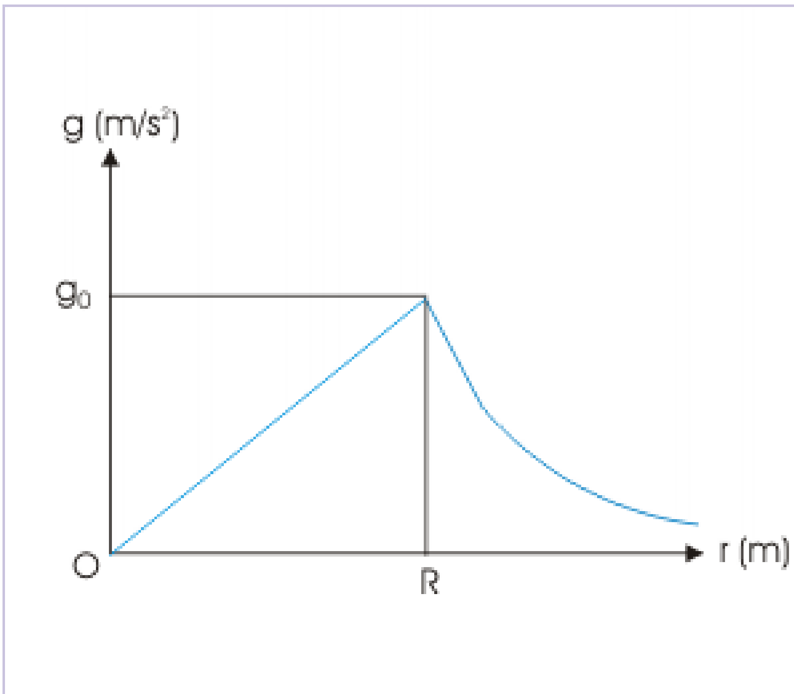

Figure 2.7: The plot shows variations in gravitational acceleration as we move away from center of Earth.

The plot above combines the effect of altitude and the effect of depth along a straight line, starting from the center of Earth.

2.3 Gravitational acceleration .vs. measured acceleration

We have made distinction between these two quantities. Here, we shall discuss the differences once again as their references and uses in problem situations can be confusing.

1: For all theoretical discussion and formulations, the idealized gravitational acceleration (g_0) is considered as a good approximation of actual gravitational acceleration on the surface of Earth, unless otherwise told. The effect of rotation is indeed a small value and hence can be neglected for all practical purposes, unless we deal with situation, requiring higher accuracy.

2: We should emphasize that both these quantities (g_0 and g) are referred to the surface of Earth. For points above or below, we use symbol (g') for effective gravitational acceleration.

3: If context requires, we should distinguish between "g0" and "g". The symbol " g_0 " denotes idealized gravitational acceleration on the surface, considering Earth (i) uniform (ii) spherical and (iii) stationary. On the other hand, "g" denotes actual measurement. We should, however, be careful to note that measured value is also not the actual measurement of gravitational acceleration. This will be clear from the point below.

4: The nature of impact of "rotation" on gravitational acceleration is different than due to other factors. We observed in our discussion in this module that "constitution of Earth" impacts the value of gravitational acceleration for a point below Earth's surface. Similarly, shape and vertical positions of measurements affect gravitational acceleration in different ways. However, these factors only account for the "actual" change in gravitational acceleration. Particularly, they do not modify the gravitational acceleration itself. For example, shape of Earth accounts for actual change in the gravitational acceleration as polar radius is actually smaller than equatorial radius.

Now, think about the change due to rotation. What does it do? It conceals a part of actual gravitational acceleration itself. A part of gravitational force is used to provide for the centripetal acceleration. We measure a different gravitational acceleration than the actual one at that point. We should keep this difference in mind while interpreting acceleration. In the nutshell, rotation alone affects measurement of actual gravitational acceleration, whereas other factors reflect actual change in gravitational acceleration.

5: What is actual gravitational acceleration anyway? From the discussion as above, it is clear that actual gravitational acceleration on the surface of Earth needs to account for the part of the gravitational force, which provides centripetal force. Hence, actual gravitational acceleration is :

$$g_{\text{actual}} = g + \omega^2 R\cos\phi$$

Note that we have made correction for centripetal force in the measured value (g) – not in the idealized value (g_0). It is so because measured value accounts actual impacts due to all factors. Hence, if we correct for rotation – which alone affects measurement of actual gravitational acceleration, then we get the actual gravitational acceleration at a point on the surface of the Earth.

Chapter 3

Gravity (application)[1]

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

3.1 Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to gravity. The questions are categorized in terms of the characterizing features of the subject matter :

- Acceleration at a Height
- Acceleration at a Depth
- Comparison of acceleration due to gravity
- Rotation of Earth
- Comparison of gravitational acceleration
- Rate of change of gravity

3.2 Acceleration at a Height

Problem 1 : At what height from the surface of Earth will the acceleration due to gravity is reduced by 36 % from the value at the surface. Take, R = 6400 km.

Solution : The acceleration due to gravity decreases as we go vertically up from the surface. The reduction of acceleration by 36 % means that the height involved is significant. As such, we can not use the approximated expression of the effective accelerations for h≪ R as given by :

$$g\prime = g\left(1 - \frac{2h}{R}\right)$$

Instead, we should use the relation,

$$\Rightarrow g\prime = \frac{g}{\left(1 + \frac{h}{R}\right)^2}$$

Note that we have considered reference gravitational acceleration equal to acceleration on the surface. Now, it is given that :

[1] This content is available online at <http://cnx.org/content/m15088/1.2/>.

$$\Rightarrow g\prime = 0.64g$$

Hence,

$$\Rightarrow 0.64g = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$

$$\Rightarrow \left(1+\frac{h}{R}\right)^2 X0.64 = 1$$

$$\Rightarrow \left(1+\frac{h}{R}\right) = \frac{10}{8} = \frac{5}{4}$$

$$\Rightarrow \frac{h}{R} = \frac{5}{4} - 1 = \frac{1}{4}$$

$$\Rightarrow h = \frac{R}{4} = \frac{6400}{4} = 1600 \quad km$$

Note : If we calculate, considering h $\ll$ R, then

$$\Rightarrow 0.64g = g\left(1 - \frac{2h}{R}\right)$$

$$\Rightarrow 0.64R = R - 2h$$

$$\Rightarrow h = \frac{R\left(1-0.64\right)}{2} = 0.18R = 0.18X6400 = 1152 \quad km$$

3.3 Acceleration at a Depth

Problem 2 : Assuming Earth to be uniform sphere, how much a weight of 200 N would weigh half way from the center of Earth.

Solution : Assuming, $g = g_0$, the accelerations at the surface (g) and at a depth (g') are related as :

$$g\prime = g\left(1 - \frac{d}{R}\right)$$

In this case,

$$\Rightarrow d = R - \frac{R}{2} = \frac{R}{2}$$

Putting in the equation of effective acceleration, we have :

$$\Rightarrow g\prime = g\left(1 - \frac{R}{2R}\right) = \frac{g}{2}$$

The weight on the surface corresponds to "mg" and its weight corresponds to "mg"'. Hence,

$$\Rightarrow mg\prime = \frac{mg}{2} = \frac{200}{2} = 100 \quad N$$

3.4 Comparison of acceleration due to gravity

Problem 3 : Find the ratio of acceleration due to gravity at a depth "h" and at a height "h" from Earth's surface. Consider h $\gg$ R, where "R" is the radius of Earth.

Solution : The acceleration due to gravity at appoint "h" below Earth's surface is given as :

$$g_1 = g_0 \left(1 - \frac{h}{R}\right)$$

The acceleration due to gravity at a point "h" above Earth's surface is given as :

$$g_2 = \frac{g_0}{\left(1 + \frac{h}{R}\right)^2}$$

Note that we have not incorporated approximation for h$\gg$R. We shall affect the same after getting the expression for the ratio .

The required ratio without approximation is :

$$\Rightarrow \frac{g_1}{g_2} = \frac{g_0 \left(1 - \frac{h}{R}\right) \left(1 + \frac{h}{R}\right)^2}{g_0}$$

$$\Rightarrow \frac{g_1}{g_2} = \left(1 - \frac{h}{R}\right) \left(1 + \frac{h}{R}\right)^2$$

$$\Rightarrow \frac{g_1}{g_2} = \left(1 - \frac{h}{R}\right) \left(1 + \frac{h^2}{R^2} + \frac{2h}{R}\right)$$

For h $\gg$ R, we can neglect terms of higher power than 1,

$$\Rightarrow \frac{g_1}{g_2} = \left(1 - \frac{h}{R}\right) \left(1 + \frac{2h}{R}\right)$$

$$\Rightarrow \frac{g_1}{g_2} = \left(1 - \frac{h}{R} + \frac{2h}{R} - \frac{2h^2}{R^2}\right)$$

Again, neglecting term with higher power,

$$\Rightarrow \frac{g_1}{g_2} = \left(1 + \frac{h}{R}\right)$$

3.5 Rotation of Earth

3.5.1

Problem 4 : If "ρ" be the uniform density of a spherical planet, then find the shortest possible period of rotation of the planet about its axis of rotation.

Solution : A planet needs to hold material it is composed. We have seen that centripetal force required for a particle on the surface is maximum at the equator. Therefore, gravitational pull of the planet should be as least sufficient enough to hold the particle at the equator. Corresponding maximum angular speed corresponding to this condition is obtained as :

$$\frac{GMm}{R^2} = m\omega^2 R$$

Time period is related to angular speed as :

$$\omega = \frac{2\pi}{T}$$

Substituting for angular speed in force equation, we get the expression involving shortest time period :

$$\Rightarrow T^2 = \frac{4\pi^2 R^3}{GM}$$

The mass of the spherical planet of uniform density is :

$$\Rightarrow M = \frac{4\pi\rho R^3}{3}$$

Putting in the equation of time period,

$$\Rightarrow T^2 = \frac{4x3\pi^2 R^3}{Gx4\pi\rho R^3} = \frac{3\pi}{G\rho}$$

$$\Rightarrow T = \sqrt{\left(\frac{3\pi}{G\rho}\right)}$$

3.5.2

Problem 5 : Considering Earth to be a sphere of uniform density, what should be the time period of its rotation about its own axis so that acceleration due to gravity at the equator becomes zero. Take g = 10 m/s^2 and R = 6400 km.

Solution : We know that the measurement of gravitational acceleration due to gravity is affected by rotation of Earth. Let g' be the effective acceleration and $g_0 = g$. Then,

$$g\prime = g - R\omega^2 \cos\Phi$$

where Φ is latitude angle.

$$\text{Here,} \quad \Phi = 0^{0,} \quad \cos\Phi = \cos 0^0 = 1,$$

$$g\prime = g - R\omega^2$$

Now, angular velocity is connected to time period as :

$$\omega = \frac{2\pi}{T}$$

Combining two equations, we have :

$$\Rightarrow g\prime = g - \frac{RX4\pi^2}{T^2}$$

$$\Rightarrow 4\pi^2 R = (g - g\prime)\, T^2$$

$$\Rightarrow T = \sqrt{\frac{4\pi^2 R}{(g - g\prime)}}$$

According to question, effective acceleration is zero,

$$g\prime = 0$$

Hence,

$$\Rightarrow T = 2\pi\sqrt{\frac{6400X10^3}{10}}$$

$$\Rightarrow T = \pi X1600 \quad s$$

$$T = 1.4 \quad hr$$

3.6 Comparison of gravitational acceleration

Problem 6 : A planet has 8 times the mass and average density that of Earth. Find acceleration on the surface of planet, considering both bodies spherical in shape. Take acceleration on the surface of Earth as 10 m/s^2.

Solution : Let subscript "1" and "2" denote Earth and planet respectively. Then, ratio of accelerations is :

$$\frac{g_2}{g_1} = \frac{\frac{GM_2}{R_2^2}}{\frac{GM_1}{R_1^2}} = \frac{M_2 R_1^2}{M_1 R_2^2}$$

Here,

$$M_2 = 8M_1$$

$$\Rightarrow \frac{g_2}{g_1} = \frac{8M_1 R_1^2}{M_1 R_2^2} = \frac{8R_1^2}{R_2^2}$$

We need to relate radii in order to evaluate the ratio as above. For this, we shall use given information about density. Here,

$$\Rightarrow \rho_2 = 8\rho_1$$

$$\Rightarrow \frac{M_2}{V_2} = \frac{8M_1}{V_1}$$

$$\text{But}, M_2 = 8M_1,$$

$$\Rightarrow \frac{8M_1}{V_2} = \frac{8M_1}{V_1}$$

$$\Rightarrow V_1 = V_2$$

$$\Rightarrow R_1 = R_2$$

Now, evaluating the ratio of accelerations, we have :

$$\Rightarrow g_2 = 8g_1 = 8X10 = 80 \quad m/s^2$$

3.7 Rate of change of gravity

3.7.1

Problem 7 : Find the rate of change of weight with respect height "h" near Earth's surface.

Solution : According to question, we are required to find the rate of change of the weight near Earth's surface. Hence, we shall use the expression for h≪R/ Also let $g_0 = g$. Then,

$$g\prime = g\left(1 - \frac{2h}{R}\right)$$

Weight at height, "h", is given by :

$$\Rightarrow W = mg\prime = mg\left(1 - \frac{2h}{R}\right) = mg - \frac{2mgh}{R}$$

The rate of change of acceleration due to gravity at a height "h" is given as :

$$\Rightarrow \frac{dW}{dh} = \frac{d}{dh}\left(mg - \frac{2mgh}{R}\right)$$

$$\Rightarrow \frac{dW}{dh} = -\frac{2mg}{R}$$

3.7.2

Problem 8 : What is fractional change in gravitational acceleration at a height "h" near the surface of Earth.

Solution : The fractional change of a quantity "x" is defined as "Δx/x". Hence, fractional change in gravitational acceleration is "Δ g/g". Let $g_0 = g$. Now, effective acceleration at a height "h" near Earth's surface is given by :

$$g\prime = g\left(1 - \frac{2h}{R}\right)$$

$$\Rightarrow g\prime - g = -\frac{2hg}{R}$$

$$\Rightarrow \frac{g\prime - g}{g} = -\frac{2h}{R}$$

$$\Rightarrow \frac{\Delta g}{g} = -\frac{2h}{R}$$

Chapter 4

Gravitational potential energy[1]

The concept of potential energy is linked to a system – not to a single particle or body. So is the case with gravitational potential energy. True nature of this form of energy is often concealed in practical consideration and reference to Earth. Gravitational energy is not limited to Earth, but is applicable to any two masses of any size and at any location. Clearly, we need to expand our understanding of various physical concepts related with gravitational potential energy.

Here, we shall recapitulate earlier discussions on potential energy and apply the same in the context of gravitational force.

4.1 Change in gravitational potential energy

The change in the gravitational potential energy of a system is related to work done by the gravitational force. In this section, we shall derive an expression to determine change in potential energy for a system of two particles. For this, we consider an elementary set up, which consists of a stationary particle of mass, " m_1 " and another particle of mass, " m_2 ", which moves from one position to another.

Now, we know that change in potential energy of the system is equal to negative of the work by gravitational force for the displacement of second particle :

$$\Delta U = -W_G$$

On the other hand, work by gravitational force is given as :

$$W_G = \int F_G dr$$

Combining two equations, the mathematical expression for determining change in potential energy of the system is obtained as :

$$\Rightarrow \Delta U = -\int_{r_1}^{r_2} F_G dr$$

In order to evaluate this integral, we need to set up the differential equation first. For this, we assume that stationary particle is situated at the origin of reference. Further, we consider an intermediate position of the particle of mass " m_2 " between two positions through which it is moved along a straight line. The change in potential energy of the system as the particle moves from position "r" to "r+dr" is :

$$dU = -F_G dr$$

[1] This content is available online at <http://cnx.org/content/m15090/1.3/>.

Change in gravitational potential energy

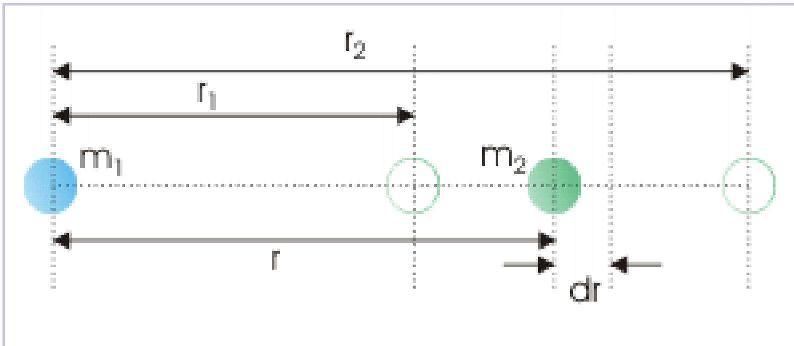

Figure 4.1: The particle is moved from one position to another.

We get the expression for the change in gravitational potential energy by integrating between initial and final positions of the second particle as :

$$\Delta U = -\int_{r_1}^{r_2} F_G dr$$

We substitute gravitational force with its expression as given by Newton's law of gravitation,

$$F = -\frac{Gm_1m_2}{r^2}$$

Note that the expression for gravitational force is preceded by a negative sign as force is directed opposite to displacement. Now, putting this value in the integral expression, we have :

$$\Rightarrow \Delta U = \int_{r_1}^{r_2} \frac{Gm_1m_2dr}{r^2}$$

Taking out constants from the integral and integrating between the limits, we have :

$$\Rightarrow \Delta U = Gm_1m_2\left[-\frac{1}{r}\right]_{r1}^{r2}$$

$$\Rightarrow \Delta U = U_2 - U_1 = Gm_1m_2\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

This is the expression of gravitational potential energy change, when a particle of mass " m_2 " moves from its position from " r_1 " to " r_2 " in the presence of particle of mass " m_1 ". It is important to realize here that " $1/r_1$ " is greater than " $1/r_2$ ". It means that the change in gravitational potential energy is positive in this case. In other words, it means that final value is greater than initial value. Hence, gravitational potential energy of the two particles system is greater for greater linear distance between particles.

4.2 Absolute gravitational potential energy

An arrangement of the system is referred to possess zero potential energy with respect to a particular reference. For this we visualize that particles are placed at very large distance. Theoretically, the conservative force like gravitation will not affect bodies which are at infinity. For this reason, zero gravitational reference potential of a system is referred to infinity. The measurement of gravitational potential energy of a system with respect to this theoretical reference is called absolute gravitational potential energy of the system.

$$U(r) = -\int_{\infty}^{r} F_G dr$$

As a matter of fact, this integral can be used to define gravitational potential energy of a system:

Definition 4.1: Gravitational potential energy

The gravitational potential energy of a system of particles is equal to "negative" of the work by the gravitational force as a particle is brought from infinity to its position in the presence of other particles of the system.

For practical consideration, we can choose real specific reference (other than infinity) as zero potential reference. Important point is that selection of zero reference is not a limitation as we almost always deal with change in potential energy – not the absolute potential energy. So long we are consistent with zero potential reference (for example, Earth's surface is considered zero gravitational potential reference), we will get the same value for the difference in potential energy, irrespective of the reference chosen.

We can also define gravitation potential energy in terms of external force as :

Definition 4.2: Gravitational potential energy

The gravitational potential energy of a system of particles is equal to the work by the external force as a particle is brought from infinity slowly to its position in the presence of other particles of the system.

4.3 Earth systems

We have already formulated expressions for gravitational potential energy for "Earth – body" system in the module on potential energy.

The potential energy of a body raised to a height "h" has been obtained as :

$$U = mgh$$

Generally, we refer gravitational potential energy of "Earth- particle system" to a particle – not to a system. This is justified on the basis of the fact that one member of the system is relatively very large in size.

All terrestrial bodies are very small with respect to massive Earth. A change in potential energy of the system is balanced by a corresponding change in kinetic energy in accordance with conservation of mechanical energy. Do we expect a change in the speed of Earth due to a change in the position of ,say, a tennis ball? All the changes due to change in the position of a tennis ball is reflected as the change in the speed of the ball itself – not in the speed of the Earth. So dropping reference to the Earth is not inconsistent to physical reality.

4.4 Gravitational potential energy of two particles system

We can determine potential energy of two particles separated by a distance "r", using the concept of zero potential energy at infinity. According to definition, the integral of potential energy of the particle is evaluated for initial position at infinity to a final position, which is at a distance "r" from the first particle at the origin of reference.

Here,

$$U_1 = 0$$

$$U_2 = U \text{ (say)}$$

$$r_1 = \infty$$

$$r_2 = r \text{ (say)}$$

Putting values in the expression of the change of potential energy, we have :

$$\Rightarrow U - 0 = Gm_1m_2\left[\frac{1}{\infty} - \frac{1}{r}\right]$$

$$\Rightarrow U = -\frac{Gm_1m_2}{r}$$

By definition, this potential energy is equal to the negative of work by gravitational force and equal to the work by an external force, which does not produce kinetic energy while the particle of mass " m_2 " is brought from infinity to a position at a distance "r" from other particle of mass " m_1 ".

We see here that gravitational potential energy is a negative quantity. As the particles are farther apart, "1/r" becomes a smaller fraction. Potential energy, being a negative quantity, increases. But, the magnitude of potential energy becomes smaller. The maximum value of potential energy is zero for r = ∞ i.e. when particles are at very large distance from each other[U+F02E]

On the other hand, the fraction "1/r" is a bigger fraction when the particles are closer. Gravitational potential energy, being a negative quantity, decreases. The magnitude of potential energy is larger. This is consistent with the fact that particles are attracted by greater force when they are closer. Hence, if a particles are closer, then it is more likely to be moved by the gravitational force. A particle away from the first particle has greater potential energy, but smaller magnitude. It is attracted by smaller gravitational force and is unlikely to be moved by gravitational force as other forces on the particle may prevail.

4.5 Gravitational potential energy of a system of particles

We have formulated expression for the gravitational potential energy of two particles system. In this section, we shall find gravitational potential energy of a system of particles, starting from the beginning. We know that zero gravitational potential energy is referred to infinity. There will no force to work with at an infinite distance. Since no force exists, no work is required for a particle to bring the first particle from infinity to a point in a gravitation free region. So the work by external force in bringing first particle in the region of zero gravitation is "zero".

Gravitational potential energy

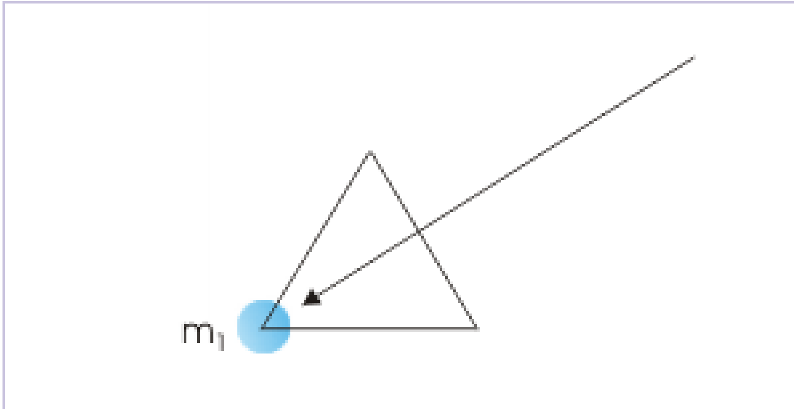

Figure 4.2: First particle is brought in a region of zero gravitation.

What about bringing the second particle (2) in the vicinity of the first particle (1)? The second particle is brought in the presence of first particle, which has certain mass. It will exert gravitational attraction on the second particle. The potential energy of two particles system will be given by the negative of work by gravitational force due to particle (1)as the second particle is brought from infinity :

Gravitational potential energy

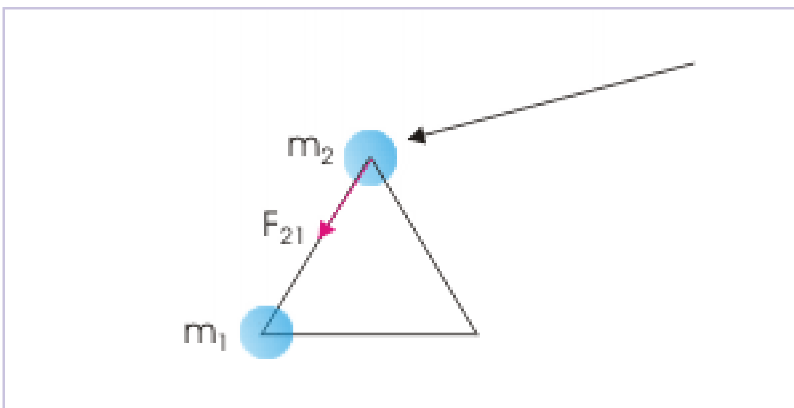

Figure 4.3: Second particle is brought in the gravitation of first particle.

$$U_{12} = -\frac{Gm_1 m_2}{r_{12}}$$

We have subscripted linear distance between first and second particle as " r_{12} ". Also note that work by gravitational force is independent of the path i.e. how force and displacement are oriented along the way second particle is brought near first particle.

Now, what about bringing the third particle of mass, " m_3 ", in the vicinity of the first two particles? The third particle is brought in the presence of first two particles, which have certain mass. They will exert gravitational forces on the third particle. The potential energy due to first particle is equal to the negative of work by gravitational force due to it :

Gravitational potential energy

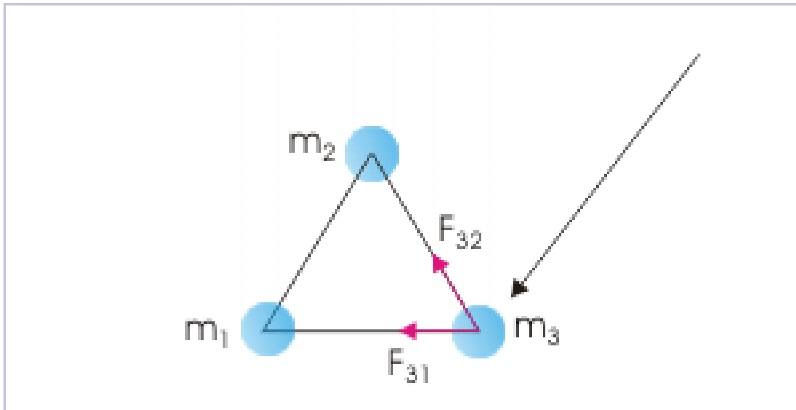

Figure 4.4: Third particle is brought in the gravitation of first and second particles.

$$U_{13} = -\frac{Gm_1 m_3}{r_{13}}$$

Similarly, the potential energy due to second particle is equal to the negative of work by gravitational force due to it :

$$U_{23} = -\frac{Gm_2 m_3}{r_{23}}$$

Thus, potential energy of three particles at given positions is algebraic sum of negative of gravitational work in (i) bringing first particle (ii) bringing second particle in the presence of first particle and (iii) bringing third particle in the presence of first two particles :

$$\Rightarrow U = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right)$$

Induction of forth particle in the system will involve work by gravitation in assembling three particles as given by the above expression plus works by the individual gravitation of three already assembled particles when fourth particle is brought from the infinity.

$$\Rightarrow U = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} + \frac{m_1 m_4}{r_{14}} + \frac{m_2 m_4}{r_{24}} + \frac{m_3 m_4}{r_{34}}\right)$$

Proceeding in this fashion, we can calculate potential energy of a system of particles. We see here that this process resembles the manner in which a system of particles like a rigid body is constituted bit by bit. As such, this potential energy of the system represents the "energy of constitution" and is called "self energy" of the rigid body or system of particles. We shall develop alternative technique (easier) to measure potential energy and hence "self energy" of regular geometric shapes with the concept of gravitational potential in a separate module.

4.5.1 Examples

Problem 1: Find the work done in bringing three particles, each having a mass of 0.1 kg from large distance to the vertices of an equilateral triangle of 10 cm in a gravity free region. Assume that no change of kinetic energy is involved in bringing particles.

Solution : We note here that all three particles have same mass. Hence, product of mass in the expression of gravitational potential energy reduces to square of mass. The gravitational potential energy of three particles at the vertices of the equilateral triangle is :

$$U = -\frac{3Gm^2}{a}$$

where "a" is the side of the equilateral triangle.

Putting values,

$$\Rightarrow U = -\frac{3X6.67X10^{-11}X0.1^2}{0.1} = -3X6.67X10^{-10}X0.01 = -20X10^{-12} \quad J$$

$$\Rightarrow U = -2X10^{-11} \quad J$$

Hence, work done by external force in bringing three particles from large distance is :

$$\Rightarrow W = U = -2X10^{-11} J$$

4.6 Work and energy

An external force working on a system brings about changes in the energy of system. If change in energy is limited to mechanical energy, then work by external force will be related to change in mechanical energy as :

$$W_F = \Delta E = \Delta U + \Delta K$$

A change in gravitational potential energy may or may not be accompanied with change in kinetic energy. It depends on the manner external force works on the system. If we work on the system in such a manner that we do not impart kinetic energy to the particles of the system, then there is no change in kinetic energy. In that case, the work by external force is equal to the change in gravitational potential energy alone.

There can be three different situations :

Case 1 : If there is change in kinetic energy, then work by external force is equal to the change in potential and kinetic energy:

$$W_F = \Delta U + \Delta K$$

Case 2 : If there is no change in kinetic energy, then work by external force is equal to the change in potential energy alone :

$$\Delta K = 0$$

Putting in the expression of work,

$$W_F = \Delta U$$

Case 3 : If there is no external force, then work by external force is zero. The change in one form of mechanical energy is compensated by a corresponding negative change in the other form. This means that mechanical energy of the system is conserved. Here,

$$W_F = 0$$

Putting in the expression of work,

$$\Rightarrow \Delta U + \Delta K = 0$$

We shall, now, work with two illustrations corresponding to following situations :

- Change in potential energy without change in kinetic energy
- Change in potential energy without external force

4.6.1 Change in potential energy without change in kinetic energy

Problem 2: Three particles, each having a mass of 0.1 kg are placed at the vertices of an equilateral triangle of 10 cm. Find the work done to change the positions of particles such that side of the triangle is 20 cm. Assume that no change of kinetic energy is involved in changing positions.

Solution : The work done to bring the particles together by external force in gravitational field is equal to potential energy of the system of particles. This means that work done in changing the positions of the particles is equal to change in potential energy due to change in the positions of particles. For work by external force,

$$W_F = \Delta U + \Delta K$$

Here, ΔK = 0

$$W_F = \Delta U$$

Now, we have seen that :

$$U = -\frac{3Gm^2}{a}$$

Hence, change in gravitational potential energy is :

$$\Rightarrow \Delta U = -\frac{3Gm^2}{a_2} - \left(-\frac{3Gm^2}{a_1}\right)$$

$$\Rightarrow \Delta U = 3Gm^2 \left[-\frac{1}{a_2} + \frac{1}{a_1}\right]$$

Putting values, we have :

$$\Rightarrow \Delta U = 3X6.67X10^{-11}X0.1^2 \left[-\frac{1}{0.2} + \frac{1}{0.1}\right]$$

$$\Rightarrow \Delta U = 3X6.67X10^{-11}X0.12X5$$

$$\Rightarrow \Delta U = 1.00X10^{-11} \quad J$$

4.6.2 Change in potential energy without external force

Problem 3: Three identical solid spheres each of mass "m" and radius "R" are released from positions as shown in the figure (assume no external gravitation). What would be the speed of any of three spheres just before they collide.

Three particles system

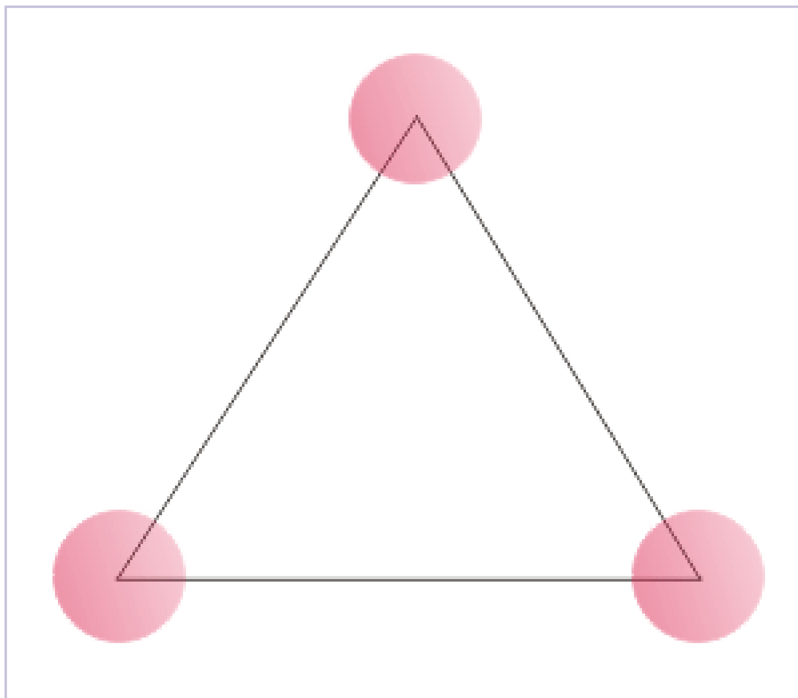

Figure 4.5: Positions before being released.

Solution : Since no external force is involved, the mechanical energy of the system at the time of release should be equal to mechanical energy just before the collision. In other words, the mechanical energy of the system is conserved. The initial potential energy of system is given by,

$$U_i = -\frac{3Gm^2}{a}$$

Let "v" be the speed of any sphere before collision. The configuration just before the collision is shown in the figure. We can see that linear distance between any two centers of two identical spheres is "2R". Hence, potential energy of the configuration before collision is,

Three particles system

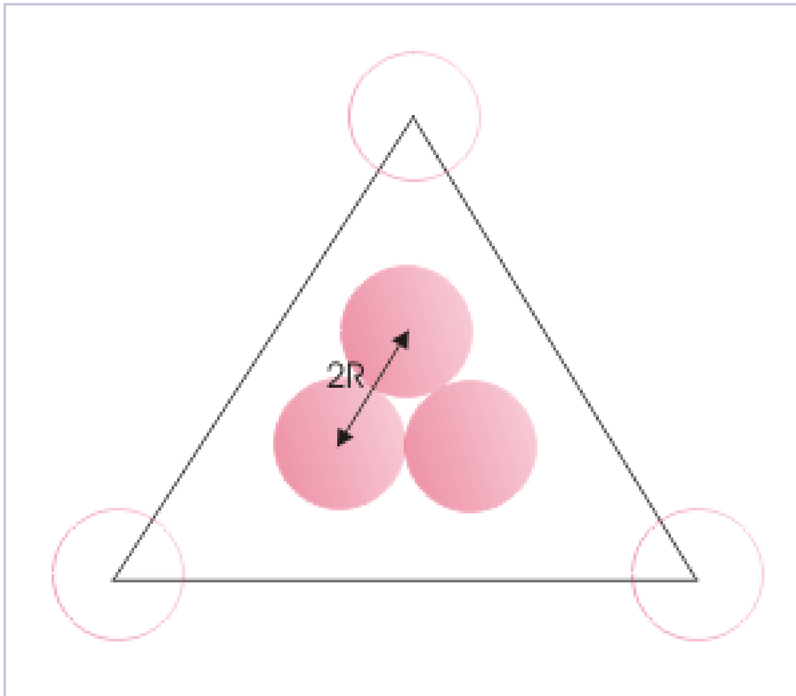

Figure 4.6: Positions just before collision.

$$U_f = -\frac{3Gm^2}{2R}$$

Applying conservation of mechanical energy,

$$K_i + U_i = K_f + U_f$$

$$\Rightarrow 0 - \frac{3Gm^2}{a} = \frac{1}{2}mv^2 - \frac{3Gm^2}{2R}$$

$$v = \sqrt{\{Gm\left(\frac{1}{R} - \frac{2}{a}\right)\}}$$

Chapter 5

Gravitational field[1]

We have studied gravitational interaction in two related manners. First, we studied it in terms of force and then in terms of energy. There is yet another way to look at gravitational interactions. We can study it in terms of gravitational field.

In the simplest form, we define a gravitational field as a region in which gravitational force can be experienced. We should, however, be aware that the concept of force field has deeper meaning. Forces like gravitational force and electromagnetic force work with "action at a distance". As bodies are not in contact, it is conceptualized that force is communicated to bodies through a force field, which operates on the entities brought in its region of influence.

Electromagnetic interaction, which also abides inverse square law like gravitational force, is completely described in terms of field concept. Theoretical conception of gravitational force field, however, is not complete yet. For this reason, we would restrict treatment of gravitational force field to the extent it is in agreement with well established known facts. In particular, we would not conceptualize about physical existence of gravitational field unless we refer "general relativity".

A body experiences gravitational force in the presence of another mass. This fact can be thought to be the result of a process in which presence of a one mass modifies the characteristics of the region around itself. In other words, it creates a gravitational field around itself. When another mass enters the region of influence, it experiences gravitational force, which is given by Newton's law of gravitation.

5.1 Field strength

Field strength (**E**) is equal to gravitational force experienced by unit mass in a gravitational field. Mathematically,

$$E = \frac{F}{m}$$

Its unit is N/kg. Field strength is a vector quantity and abides by the rules of vector algebra, including superposition principle. Hence, if there are number of bodies, then resultant or net gravitational field due to them at a given point is vector sum of individual fields,

$$E = E_1 + E_2 + E_3 + \ldots\ldots.$$

$$\Rightarrow E = \Sigma E_i$$

[1]This content is available online at <http://cnx.org/content/m15091/1.5/>.

5.2 Significance of field strength

An inspection of the expression of gravitational field reveals that its expression is exactly same as that of acceleration of a body of mass, “m”, acted upon by an external force, “**F**”. Clearly,

$$E = a = \frac{F}{m}$$

For this reason, gravitational field strength is dimensionally same as acceleration. Now, dropping vector notation for action in a particular direction of force,

$$\Rightarrow E = a = \frac{F}{m}$$

We can test this assertion. For example, Earth’s gravitational field strength can be obtained, by substituting for gravitational force between Earth of mass,"M", and a particle of mass, "m" :

$$\Rightarrow E = \frac{F}{m} = \frac{GMm}{r^2 m} = \frac{GM}{r^2} = g$$

Thus, Earth’s gravitational field strength is equal to gravitational acceleration, “g”.

Field strength, apart from its interpretation for the action at a distance, is a convenient tool to map a region and thereby find the force on a body brought in the field. It is something like knowing “unit rate”. Suppose if we are selling pens and if we know its unit selling price, then it is easy to calculate price of any numbers of pens that we sale. We need not compute the unit selling price incorporating purchase cost, overheads, profit margins etc every time we make a sale.

Similar is the situation here. Once gravitational field strength in a region is mapped (known), we need not be concerned about the bodies which are responsible for the gravitational field. We can compute gravitational force on any mass that enters the region by simply multiplying the mass with the unit rate of gravitational force i.e. field strength,

$$F = mE$$

In accordance with this interpretation, we determine gravitational force on a body brought in the gravitational field of Earth by multiplying the mass with the gravitational field strength,

$$\Rightarrow F = mE = mg$$

This approach has following advantages :

1: We can measure gravitational force on a body without reference to other body responsible for gravitational field. In the context of Earth, for example, we compute gravitational force without any reference to the mass of Earth. The concept of field strength allows us to study gravitational field in terms of the mass of one body and as such relieves us from considering it always in terms of two body system. The effect of one of two bodies is actually represented by its gravitational field strength.

2: It simplifies mathematical calculation for gravitational force. Again referring to the context of Earth’s gravity, we see that we hardly ever use Newton’s gravitational law. We find gravitational force by just multiplying mass with gravitational field strength (acceleration). Imagine if we have to compute gravitational force every time, making calculation with masses of Earth and the body and the squared distance between them!

5.2.1 Comparison with electrostatic field

There is one very important aspect of gravitational field, which is unique to it. We can appreciate this special feature by comparing gravitational field with electrostatic field. We know that the electrostatic force, like gravitational force, also follows inverse square law. Electrostatic force for two point charges separated by a linear distance, "r", is given by Coulomb’s law as :

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}$$

The electrostatic field (E_E)is defined as the electrostatic force per unit positive charge and is expressed as :

$$E_E = \frac{F_E}{q} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}q = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

The important point, here, is that electrostatic field is not equal to acceleration. Recall that Newton's second law of motion connects force (any type) with "mass" and "acceleration" as :

$$F = ma$$

This relation is valid for all kinds of force - gravitational or electrostatic or any other type. What we mean to say that there is no corresponding equation like "F=qa". Mass only is the valid argument of this relation. As such, electrostatic field can not be equated with acceleration as in the case of gravitational field.

Thus, equality of "field strength" with "acceleration" is unique and special instance of gravitational field - not a common feature of other fields. As a matter of fact, this instance has a special significance, which is used to state "equivalence of mass" - the building block of general theory of relativity.

We shall discuss this concept in other appropriate context. Here, we only need to underline this important feature of gravitational field.

5.2.2 Example

Problem 1: A charged particle of mass "m" carries a charge "q". It is projected upward from Earth's surface in an electric field "E", which is directed downward. Determine the nature of potential energy of the particle at a given height, "h".

Solution : The charged particle is acted upon simultaneously by both gravitational and electrostatic fields. Here, gravity works against displacement. The work by gravity is, therefore, negative. Hence, potential energy arising from gravitational field (with reference from surface) is positive as :

$$U_G = -W_G = -\left(-F_G h\right) = mE_G h = mgh$$

As given in the question, the electrostatic field is acting downward. Since charge on the particle is positive, electrostatic force acts downward. It means that work by electrostatic force is also negative. Hence, potential energy arising from electrostatic field (with reference from surface) is :

$$U_E = -W_E = -\left(-F_E h\right) = qE_E h = qEh$$

Total potential energy of the charged particle at a height "h" is :

$$\Rightarrow U = U_G + U_E = (mgh + qEh) = (mg + qE)\,h$$

The quantities in the bracket are constant. Clearly, potential energy is a function of height.

It is important to realize that description in terms of respective fields enables us to calculate forces without referring to either Newton's gravitation law or Coulomb's law of electrostatic force.

5.3 Gravitational field due to a point mass

Determination of gravitational force strength due to a point mass is easy. It is so because, Newton's law of gravitation provides the expression for determining force between two particles.

Let us consider a particle of mass, "M", for which we are required to find gravitational field strength at a certain point, "P". For convenience, let us consider that the particle is situated at the origin of the reference system. Let the point, where gravitational field is to be determined, lies at a distance "r" from the origin on the reference line.

Gravitational field strength

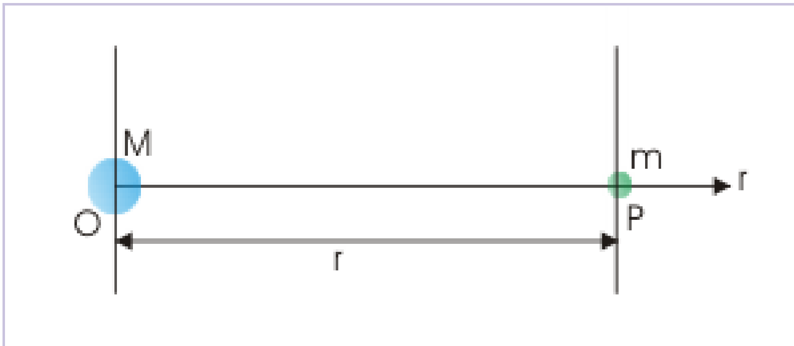

Figure 5.1: Gravitational field at a point "P" due to mass "M"

We should make it a point to understand that the concept of gravitational field is essentially "one" particle/ body/entity concept. We need to measure gravitational force at the point, "P", on a unit mass as required by the definition of field strength. It does not exist there. In order to determine field strength, however, we need to visualize as if unit mass is actually present there.

We can do this two ways. Either we visualize a point mass exactly of unit value or we visualize any mass, "m", and then calculate gravitational force. In the later case, we divide the gravitational force as obtained from Newton's law of gravitation by the mass to get the force per unit mass. In either case, we call this point mass as test mass. If we choose to use a unit mass, then :

$$\Rightarrow E = F = \frac{GMX1}{r^2} = \frac{GM}{r^2}$$

On the other hand, if we choose any arbitrary test mass, "m", then :

$$\Rightarrow E = \frac{F}{m} = \frac{GMm}{r^2 m} = \frac{GM}{r^2}$$

However, there is a small catch here. The test mass has its own gravitational field. This may unduly affect determination of gravitational field due to given particle. In order to completely negate this possibility, we may consider a mathematical expression as given here, which is more exact for defining gravitational field :

$$E = \lim_{m \to 0} \frac{F}{m}$$

Nevertheless, we know that gravitational force is not a very strong force. The field of a particle of unit mass can safely be considered negligible.

The expression for the gravitational field at point "P", as obtained above, is a scalar value. This expression, therefore, measures the magnitude of gravitational field - not its direction. We can realize from the figure shown above that gravitational field is actually directed towards origin, where the first particle is situated. This direction is opposite to the positive reference direction. Hence, gravitational field strength in vector form is preceded by a negative sign :

$$\Rightarrow E = \frac{F}{m} = -\frac{GM}{r^2}\hat{r}$$

where " $\hat{r}$ " is unit vector in the reference direction.

The equation obtained here for the gravitational field due to a particle of mass, "M", is the basic equation for determining gravitational field for any system of particles or rigid body. The general idea is to consider the system being composed of small elements, each of which can be treated at particle. We, then, need to find the net or resultant field, following superposition principle. We shall use this technique to determine gravitational field due to certain regularly shaped geometric bodies in the next module.

5.4 Example

Problem 2 : The gravitational field in a region is in xy-plane is given by $\mathbf{3i} + \mathbf{j}$. A particle moves along a straight line in this field such that work done by gravitation is zero. Find the slope of straight line.

Solution : The given gravitational field is a constant field. Hence, gravitational force on the particle is also constant. Work done by a constant force is given as :

$$W = F.r$$

Let "m" be the mass of the particle. Then, work is given in terms of gravitational field as :

$$\Rightarrow W = mE.r$$

Work done in the gravitational field is zero, if gravitational field and displacement are perpendicular to each other. If " s_1 " and " s_2 " be the slopes of the direction of gravitational field and that of straight path, then the slopes of two quantities are related for being perpendicular as :

Work by gravitational force

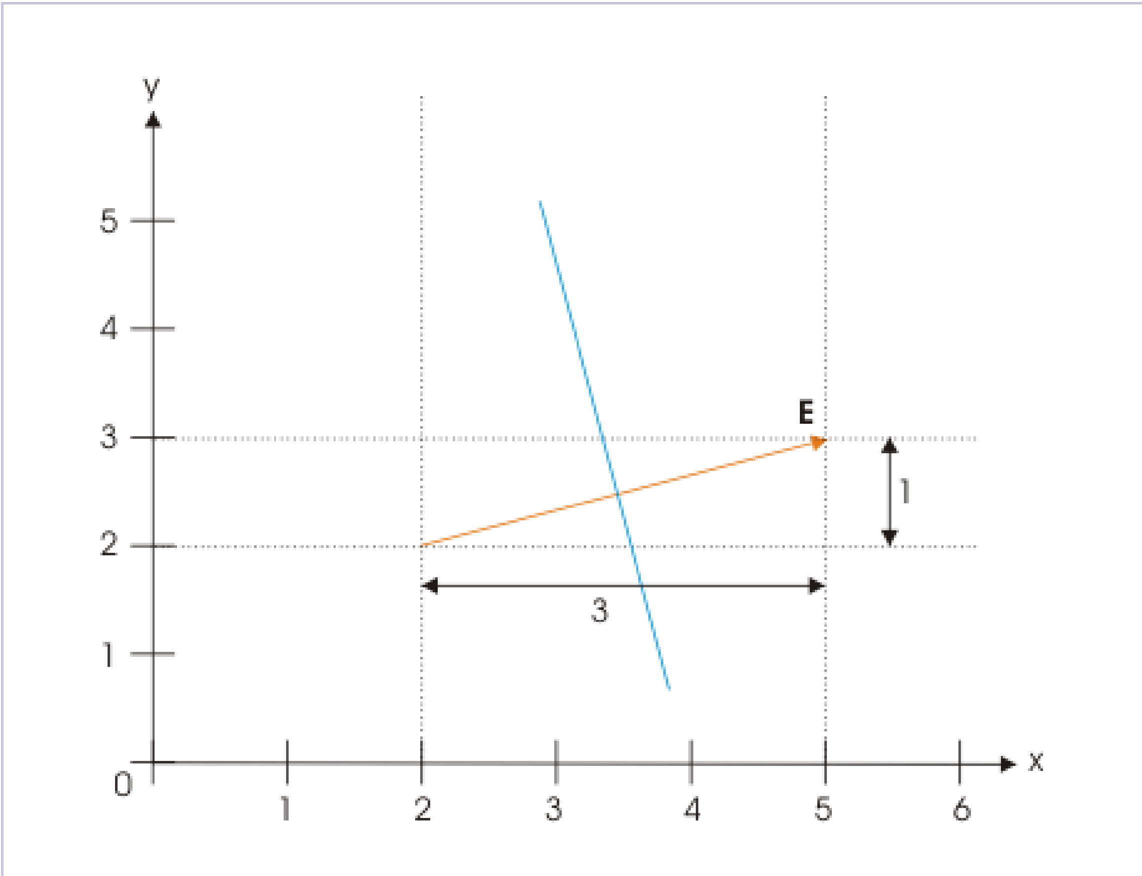

Figure 5.2: Gravitational field and displacement of particle are perpendicular to each other.

$$s_1 s_2 = -1$$

Note that slope of a straight line is usually denoted by letter "m". However, we have used letter "s" in this example to distinguish it from mass, which is also represented by letter "m".

In order to find the slope of displacement, we need to know the slope of the straight line, which is perpendicular to the direction of gravitational field.

Now, the slope of the line of action of gravitational field is :

$$\Rightarrow s_1 = \frac{1}{3}$$

Hence, for gravitational field and displacement to be perpendicular,

$$\Rightarrow s_1 s_2 = \left(\frac{1}{3}\right) s_2 = -1$$

$$s_2 = -3$$

Chapter 6

Gravitational field due to rigid bodies[1]

6.1 Gravitational field of rigid bodies

We shall develop few relations here for the gravitational field strength of bodies of particular geometric shape without any reference to Earth's gravitation.

Newton's law of gravitation is stated strictly in terms of point mass. The expression of gravitational field due to a particle, as derived from this law, serves as starting point for developing expressions of field strength due to rigid bodies. The derivation for field strength for geometric shapes in this module, therefore, is based on developing technique to treat a real body mass as aggregation of small elements and combine individual effects. There is a bit of visualization required as we need to combine vectors, having directional property.

Along these derivations for gravitational field strength, we shall also establish Newton's shell theory, which has been the important basic consideration for treating spherical mass as point mass.

The celestial bodies - whose gravitational field is appreciable and whose motions are subject of great interest - are usually spherical. Our prime interest, therefore, is to derive expression for field strength of solid sphere. Conceptually, a solid sphere can be considered being composed of infinite numbers of closely packed spherical shells. In turn, a spherical shell can be conceptualized to be aggregation of thin circular rings of different diameters.

The process of finding the net effect of these elements fits perfectly well with integration process. Our major task, therefore, is to suitably set up an integral expression for elemental mass and then integrate the elemental integral between appropriate limits. It is clear from the discussion here that we need to begin the process in the sequence starting from ring –> spherical shell –> solid sphere.

6.1.1 Gravitational field due to a uniform circular ring

We need to find gravitational field at a point "P" lying on the central axis of the ring of mass "M" and radius "a". The arrangement is shown in the figure. We consider a small mass "dm" on the circular ring. The gravitational field due to this elemental mass is along PA. Its magnitude is given by :

[1] This content is available online at <http://cnx.org/content/m15104/1.2/>.

Gravitational field due to a ring

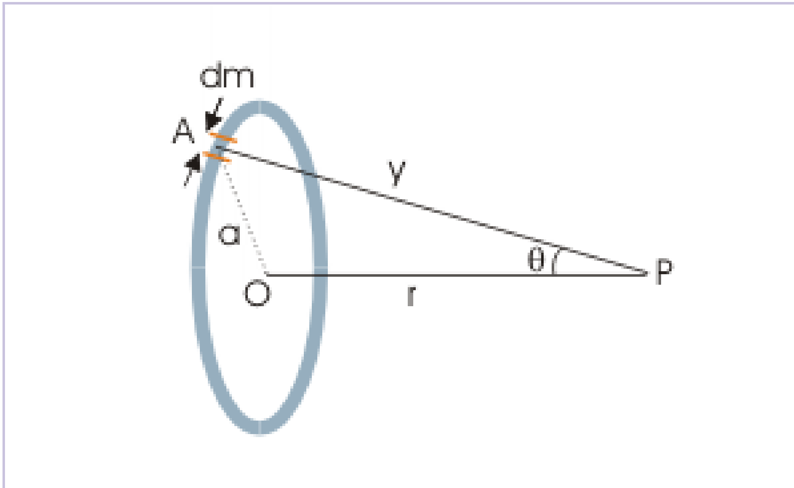

Figure 6.1: The gravitational field is measured on axial point "P".

$$dE = \frac{Gdm}{PA^2} = \frac{Gdm}{(a^2 + r^2)}$$

We resolve this gravitational field in the direction parallel and perpendicular to the axis in the plane of OAP.

Gravitational field due to a ring

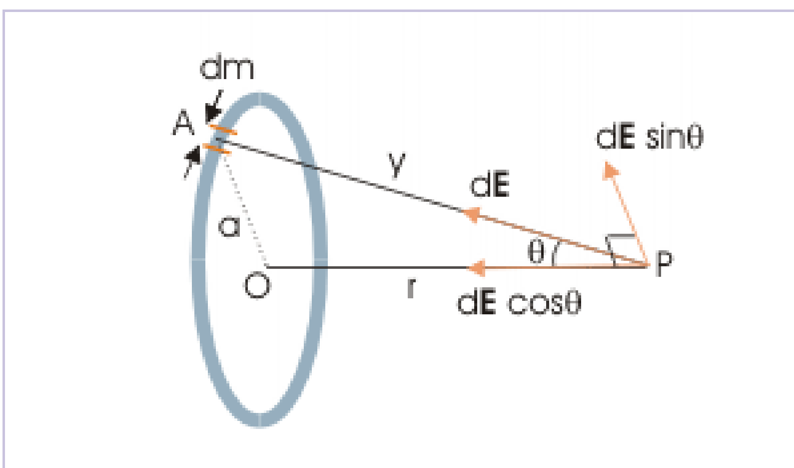

Figure 6.2: The net gravitational field is axial.

$$dE_{||} = dE\cos\theta$$

$$dE_{\perp} = dE\sin\theta$$

We note two important things. First, we can see from the figure that measures of "y" and "θ" are same for all elemental mass. Further, we are considering equal elemental masses. Therefore, the magnitude of gravitational field due any of the elements of mass "dm" is same, because they are equidistant from point "P".

Second, perpendicular components of elemental field intensity for pair of elemental masses on diametrically opposite sides of the ring are oppositely directed. On integration, these perpendicular components will add up to zero for the whole of ring. It is clear that we can assume zero field strength perpendicular to axial line, if mass distribution on the ring is uniform. For uniform ring, the net gravitational intensity will be obtained by integrating axial components of elemental field strength only. Hence,

Gravitational field due to a ring

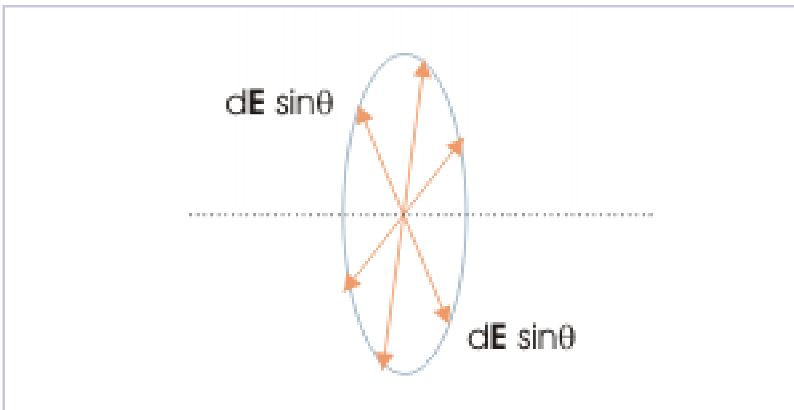

Figure 6.3: Perpendicular components cancel each other.

$$\Rightarrow E = \int dE\cos\theta$$

$$\Rightarrow E = \int \frac{Gdm\cos\theta}{(a^2 + r^2)}$$

The trigonometric ratio "cosθ" is a constant for all points on the ring. Taking out cosine ratio and other constants from the integral,

$$\Rightarrow E = \frac{G\cos\theta}{(a^2 + r^2)} \int dm$$

Integrating for m = 0 to m = M, we have :

$$\Rightarrow E = \frac{GM\cos\theta}{(a^2 + r^2)}$$

From triangle OAP,

$$\Rightarrow \cos\theta = \frac{r}{(a^2 + r^2)^{\frac{1}{2}}}$$

Substituting for "cosθ" in the equation ,

$$\Rightarrow E = \frac{GMr}{(a^2 + r^2)^{\frac{3}{2}}}$$

For r = 0, E = 0. The gravitation field at the center of ring is zero. This result is expected also as gravitational fields due to two diametrically opposite equal elemental mass are equal and opposite and hence balances each other.

6.1.1.1 Position of maximum gravitational field

We can get the maximum value of gravitational field by differentiating its expression w.r.t linear distance and equating the same to zero,

$$\frac{dE}{dr} = 0$$

This yields,

$$\Rightarrow r = \frac{a}{\sqrt{2}}$$

Substituting in the expression of gravitational field, the maximum field strength due to a circular ring is :

$$\Rightarrow E_{\max} = \frac{GMa}{2^{\frac{1}{2}}\left(a^2 + \frac{a^2}{2}\right)^{\frac{3}{2}}} = \frac{GMa}{3\sqrt{3}a^2}$$

The plot of gravitational field with axial distance shows the variation in the magnitude,

Gravitational field due to a ring

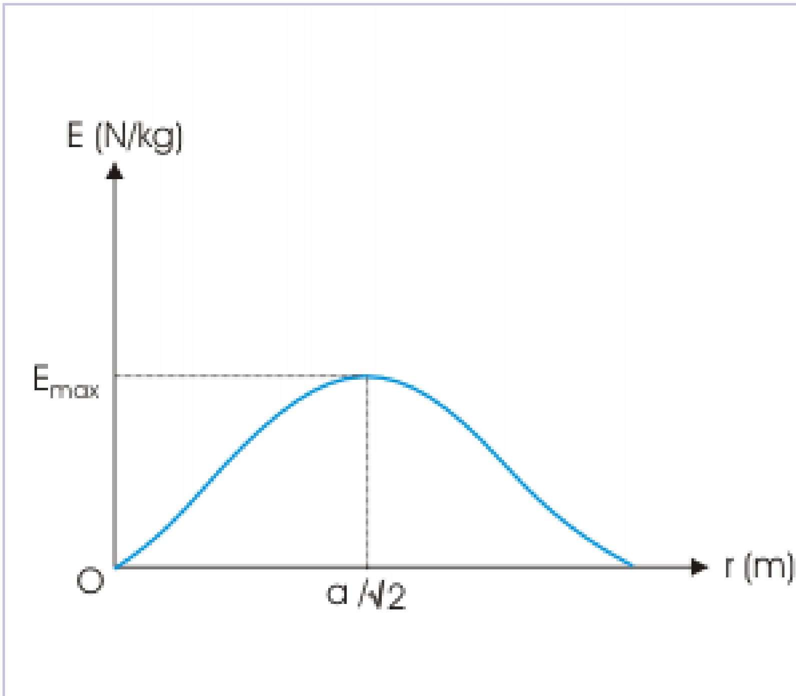

Figure 6.4: The gravitational field along the axial line.

6.1.2 Gravitational field due to thin spherical shell

The spherical shell of radius "a" and mass "M" can be considered to be composed of infinite numbers of thin rings. We consider one such ring of infinitesimally small thickness "dx" as shown in the figure. We derive the required expression following the sequence of steps as outlined here :

Gravitational field due to thin spherical shell

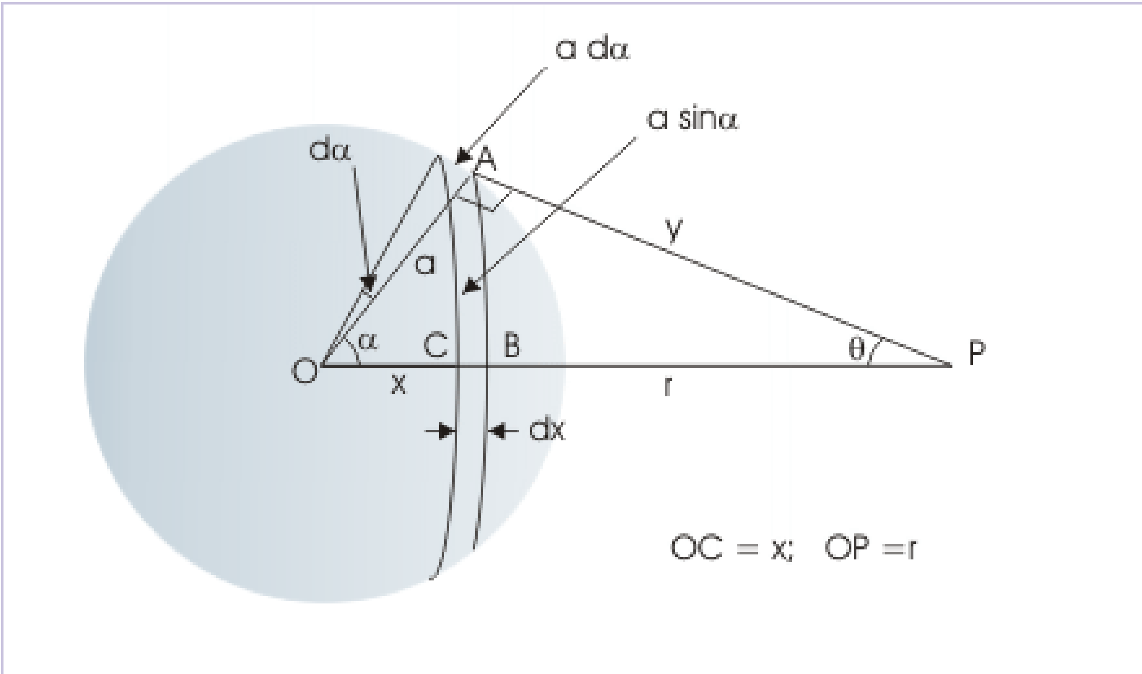

Figure 6.5: The gravitational field is measured on axial point "P".

(i) Determine mass of the elemental ring in terms of the mass of shell and its surface area.

$$dm = \frac{M}{4\pi a^2} X 2\pi a \sin\alpha dx = \frac{M a \sin\alpha dx}{2a^2}$$

From the figure, we see that :

$$dx = a d\alpha$$

Putting these expressions,

$$\Rightarrow dm = \frac{M a \sin\alpha dx}{2a^2} = \frac{M a \sin\alpha a d\alpha}{2a^2} = \frac{M \sin\alpha d\alpha}{2}$$

(ii) Write expression for the gravitational field due to the elemental ring. For this, we employ the formulation derived earlier for the ring,

$$\Rightarrow dE = \frac{G dm \cos\theta}{AP^2}$$

Putting expression for elemental mass,

$$\Rightarrow dE = \frac{G M \sin\alpha d\alpha \cos\theta}{2y^2}$$

(v) Set up integral for the whole disc

We see here that gravitational fields due to all concentric rings are directed towards the center of spherical shell along the axis.

$$\Rightarrow E = GM \int \frac{\sin\alpha\cos\theta d\alpha}{2y^2}$$

The integral expression has three varibles "α","θ" and "y".Clearly, we need to express variables in one variable "x". From triangle, OAP,

$$\Rightarrow y^2 = a^2 + r^2 - 2ar\cos\alpha$$

Differentiating each side of the equation,

$$\Rightarrow 2ydy = 2ar\sin\alpha d\alpha$$

$$\Rightarrow \sin\alpha d\alpha = \frac{ydy}{ar}$$

Again from triangle OAP,

$$\Rightarrow a^2 = y^2 + r^2 - 2yr\cos\theta$$

$$\Rightarrow \cos\theta = \frac{y^2 + r^2 - a^2}{2yr}$$

Putting these values in the integral,

$$\Rightarrow E = GM \int \frac{dy\left(y^2 + r^2 - a^2\right)}{4ar^2y^2}$$

$$\Rightarrow E = GM \int \frac{dy}{4ar^2}\left(1 - \frac{a^2 - r^2}{y^2}\right)$$

We shall decide limits of integration on the basis of the position of point "P" – whether it lies inside or outside the shell. Integrating expression on right side between two general limits, initial (L_1) and final (L_2),

$$\Rightarrow E = GM \int_{L_1}^{L_2} \frac{dy}{4ar^2}\left(1 - \frac{a^2 - r^2}{y^2}\right)$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[y + \frac{a^2 - r^2}{y}\right]_{L1}^{L2}$$

6.1.2.1 Evaluation of integral for the whole shell

Case 1 : The point "P" lies outside the shell. The total gravitational field is obtained by integrating the integral from y = r-a to y = r+a,

$$\Rightarrow E = \frac{GM}{4ar^2}\left[y + \frac{a^2 - r^2}{y}\right]_{r-a}^{r+a}$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[r + a + \frac{a^2 - r^2}{r + a} - r + a - \frac{a^2 - r^2}{r - a}\right]$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[2a + \left(a^2 - r^2\right)\left(\frac{1}{r + a} - \frac{1}{r - a}\right)\right]$$

$$\Rightarrow E = \frac{GM}{4ar^2} X 4a$$

$$\Rightarrow E = \frac{GM}{r^2}$$

This is an important result. We have been using this result by the name of Newton's shell theory. According to this theory, a spherical shell, for a particle outside it, behaves as if all its mass is concentrated at its center. This is how we could calculate gravitational attraction between Earth and an apple. Note that radius of the shell, "a", does not come into picture.

Case 2 : The point "P" lies outside the shell. The total gravitational field is obtained by integrating the integral from x = a-r to x = a+r,

$$\Rightarrow E = \frac{GM}{4ar^2}\left[y + \frac{a^2 - r^2}{y}\right]_{a-r}^{a+r}$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[a + r + \frac{a^2 - r^2}{a + r} - a + r - \frac{a^2 - r^2}{a - r}\right]$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[2r + \left(a^2 - r^2\right)\left(\frac{1}{a + r} - \frac{1}{a - r}\right)\right]$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[2r - 2r\right] = 0$$

This is yet another important result, which has been used to determine gravitational acceleration below the surface of Earth. The mass residing outside the sphere drawn to include the point below Earth's surface, does not contribute to gravitational force at that point.

The mass outside the sphere is considered to be composed of infinite numbers of thin shells. The point within the Earth lies inside these larger shells. As gravitational intensity is zero within a shell, the outer shells do not contribute to the gravitational force on the particle at that point.

A plot, showing the gravitational field strength, is shown here for regions both inside and outside spherical shell :

Gravitational field due to thin spherical shell

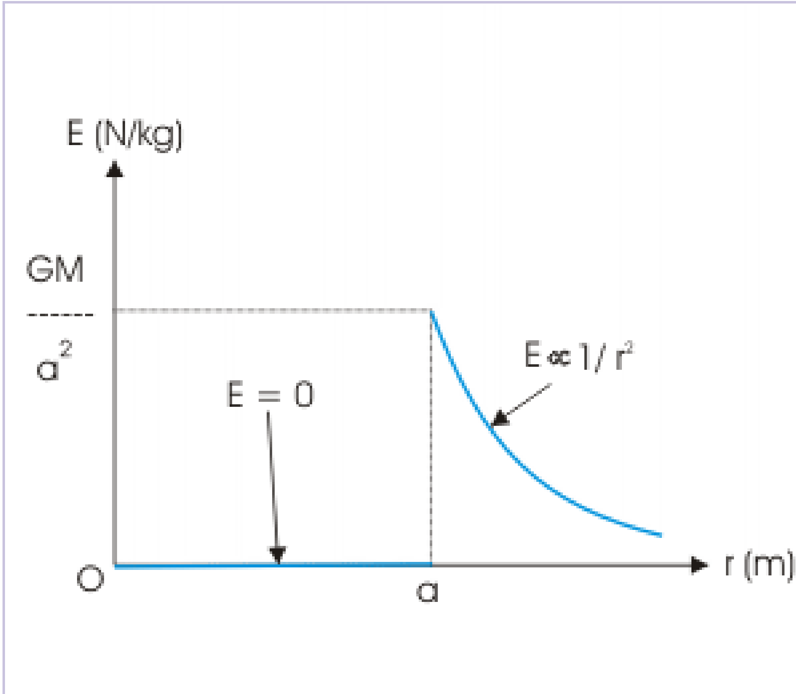

Figure 6.6: The gravitational field along linear distance from center.

6.1.3 Gravitational field due to uniform solid sphere

The uniform solid sphere of radius "a" and mass "M" can be considered to be composed of infinite numbers of thin spherical shells. We consider one such spherical shell of infinitesimally small thickness "dx" as shown in the figure. The gravitational field strength due to thin spherical shell at a point outside shell, which is at a linear distance "r" from the center, is given by

Gravitational field due to solid sphere

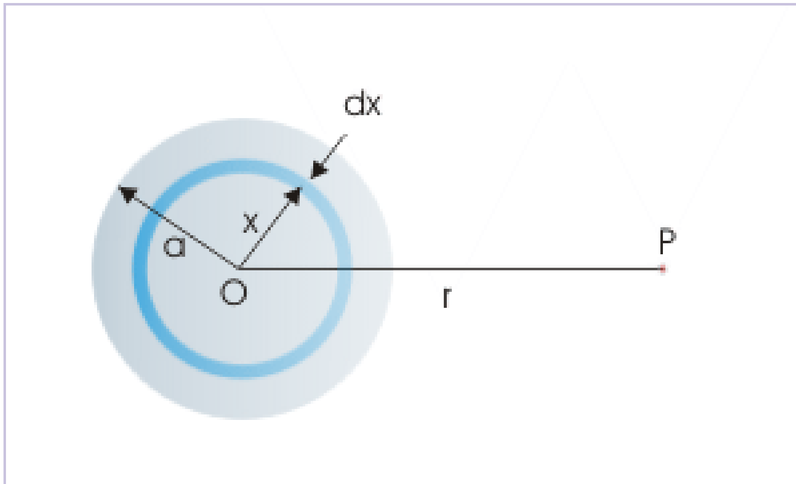

Figure 6.7: The gravitational field at a distance "r" from the center of sphere.

$$dE = \frac{Gdm}{r^2}$$

The gravitational field strength acts along the line towards the center of sphere. As such, we can add gravitational field strengths of individual shells to obtain the field strength of the sphere. In this case, most striking point is that the centers of all spherical shells are coincident at one point. This means that linear distance between centers of spherical shell and the point ob observation is same for all shells. In turn, we can conclude that the term " r^2 " is constant for all spherical shells and as such can be taken out of the integral,

$$\Rightarrow E = \int \frac{Gdm}{r^2} = \frac{G}{r^2} \int dm = \frac{GM}{r^2}$$

We can see here that a uniform solid sphere behaves similar to a shell. For a point outside, it behaves as if all its mass is concentrated at its center. Note that radius of the sphere, "a", does not come into picture. Sphere behaves as a point mass for a point outside.

6.1.3.1 Gravitational field at an inside point

We have already derived this relation in the case of Earth.

For this reason, we will not derive this relation here. Nevertheless, it would be intuitive to interpret the result obtained for the acceleration (field strength) earlier,

Gravitational field inside solid sphere

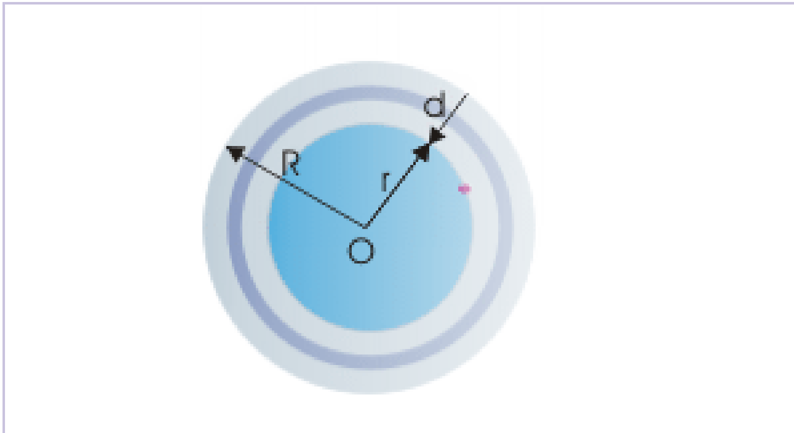

Figure 6.8: The gravitational field at a distance "r" from the center of sphere.

$$\Rightarrow g\prime = g_0 \left(1 - \frac{d}{R}\right)$$

Putting value of "g0" and simplifying,

$$\Rightarrow g\prime = \frac{GM}{R^2}\left(1 - \frac{d}{R}\right) = \frac{GM}{R^2}\left(\frac{R-d}{R}\right) = \frac{GMr}{R^3}$$

As we have considered "a" as the radius of sphere here – not "R" as in the case of Earth, we have the general expression for the field strength insider a uniform solid sphere as :

$$\Rightarrow E = \frac{GMr}{a^3}$$

The field strength of uniform solid sphere within it decreases linearly within "r" and becomes zero as we reach at the center of the sphere. A plot, showing the gravitational field strength, is shown here for regions both inside and outside :

Gravitational field due to uniform solid sphere

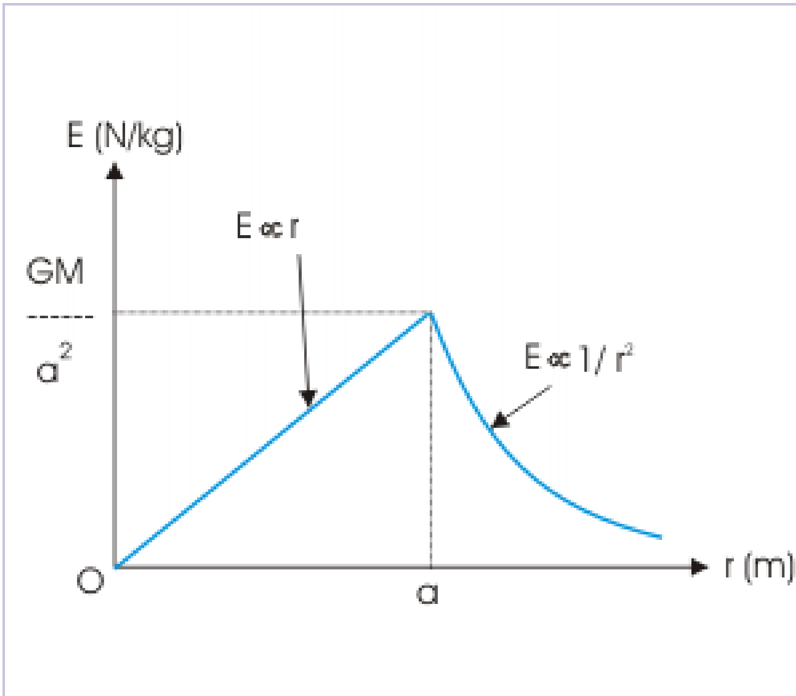

Figure 6.9: The gravitational field along linear distance from center.

Chapter 7

Gravitational field (application)[1]

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

7.1 Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to gravitational field. The questions are categorized in terms of the characterizing features of the subject matter :

- Gravitational field
- Gravitational force
- Superposition principle

7.2 Gravitational field

Problem 1 : Calculate gravitational field at a distance "r" from the center of a solid sphere of uniform density, "ρ", and radius "R". Given that $r < R$.

[1] This content is available online at <http://cnx.org/content/m15106/1.1/>.

Gravitational field

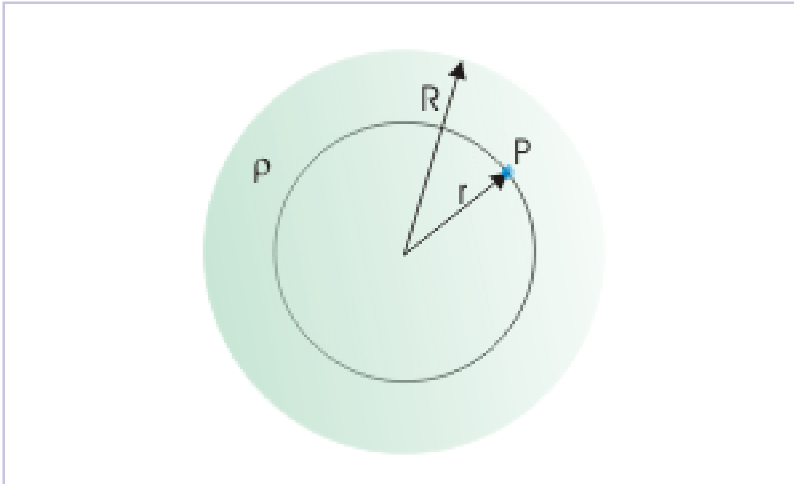

Figure 7.1: Gravitational field inside a solid sphere.

Solution : The point is inside the solid sphere of uniform density. We apply the theorem that gravitational field due to mass outside the sphere of radius "r" is zero at the point where field is being calculated. Let the mass of the sphere of radius "r" be "m", then :

$$m = \frac{4}{3}\pi r^3 \rho$$

The gravitational field due to this sphere on its surface is given by :

$$E = \frac{Gm}{r^2} = \frac{GX\frac{4}{3}\pi r^3 \rho}{r^2}$$

$$\Rightarrow E = \frac{4G\pi r\rho}{3}$$

7.3 Gravitational force

Problem 2 : A sphere of mass "2M" is placed a distance "$\sqrt{3}$ R" on the axis of a vertical ring of radius "R" and mass "M". Find the force of gravitation between two bodies.

Gravitational force

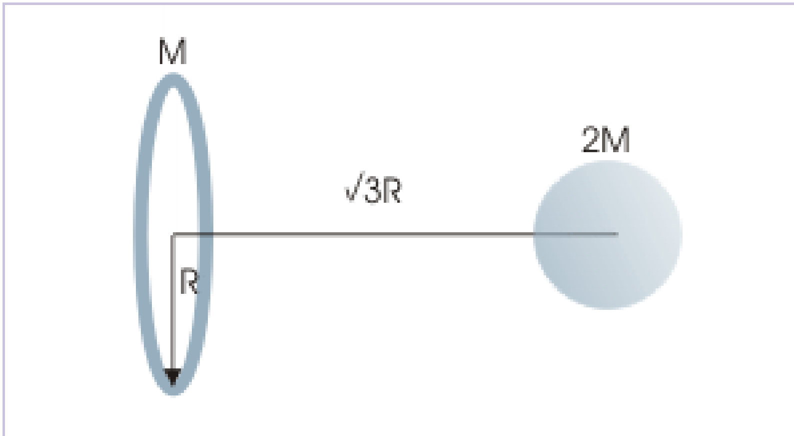

Figure 7.2: The center of sphere lies on the axis of ring.

Solution : Here, we determine gravitational field due to ring at the axial position, where center of sphere lies. Then, we multiply the gravitational field with the mass of the sphere to calculate gravitational force between two bodies.

The gravitational field due to ring on its axis is given as :

$$E = \frac{GMx}{\left(R^2 + x^2\right)^{\frac{3}{2}}}$$

Putting values,

$$\Rightarrow E = \frac{GM\sqrt{3}R}{\left\{R^2 + \left(\sqrt{3}R\right)^2\right\}^{\frac{3}{2}}}$$

$$\Rightarrow E = \frac{\sqrt{3}GM}{8R^2}$$

The sphere acts as a point mass. Therefore, the gravitational force between two bodies is :

$$\Rightarrow F = 2ME = \frac{2\sqrt{3}GM^2}{8R^2} = \frac{\sqrt{3}GM^2}{4R^2}$$

7.4 Superposition principle

7.4.1

Problem 3 : A spherical cavity is made in a solid sphere of mass "M" and radius "R" as shown in the figure. Find the gravitational field at the center of cavity due to remaining mass.

Superposition principle

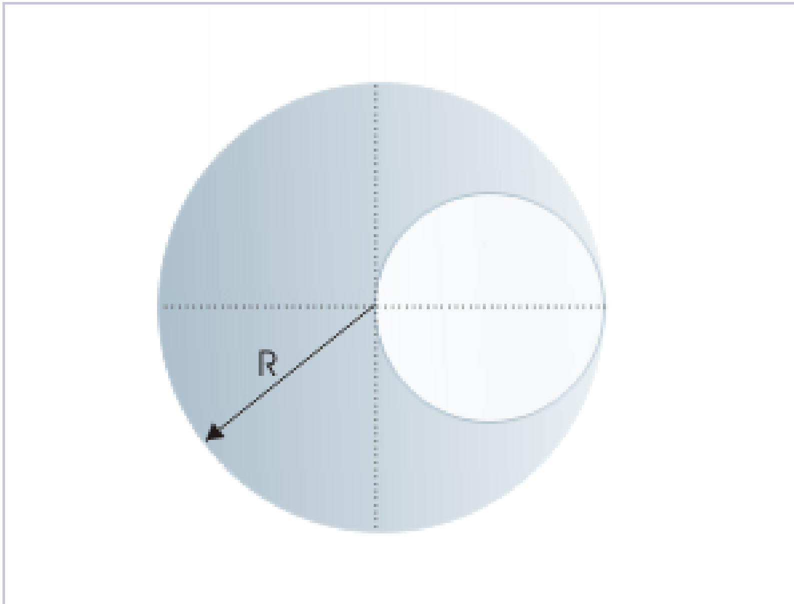

Figure 7.3: The gravitational field at the center of spherical cavity

Solution : According to superposition principle, gravitational field (**E**) due to whole mass is equal to vector sum of gravitational field due to remaining mass (E_1) and removed mass (E_2).

$$E = E_1 + E_2$$

The gravitation field due to a uniform solid sphere is zero at its center. Therefore, gravitational field due to removed mass is zero at its center. It means that gravitational field due to solid sphere is equal to gravitational field due to remaining mass. Now, we know that "**E**" at the point acts towards center of sphere. As such both "**E**" and " E_1 " acts along same direction. Hence, we can use scalar form,

$$E_1 = E$$

Now, gravitational field due to solid sphere of radius "R" at a point "r" within the sphere is given as :

$$E = \frac{GMr}{R^3}$$

Here,

Superposition principle

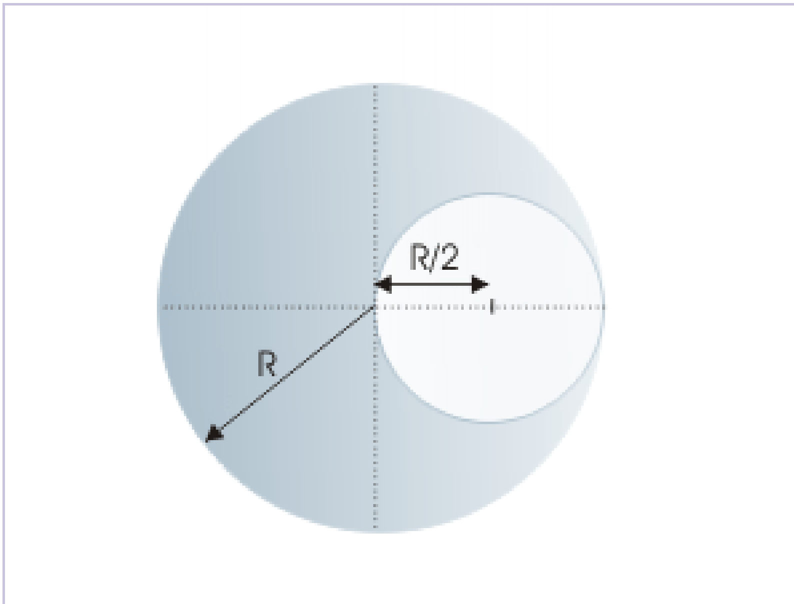

Figure 7.4: The gravitational field at the center of spherical cavity

$$r = R - \frac{R}{2} = \frac{R}{2}$$

Thus,

$$\Rightarrow E = \frac{GMR}{2R^3} = \frac{GM}{2R^2}$$

Therefore, gravitational field due to remaining mass, " E_1 ", is :

$$\Rightarrow E_1 = E = \frac{GM}{2R^2}$$

7.4.2

Problem 4 : Two concentric spherical shells of mass " m_1 " and " m_2 " have radii " r_1 " and " r_2 " respectively, where $r_2 > r_1$. Find gravitational intensity at a point, which is at a distance "r" from the common center for following situations, when it lies (i) inside smaller shell (ii) in between two shells and (iii) outside outer shell.

Solution : Three points "A", "B" and "C" corresponding to three given situations in the question are shown in the figure :

Superposition principle

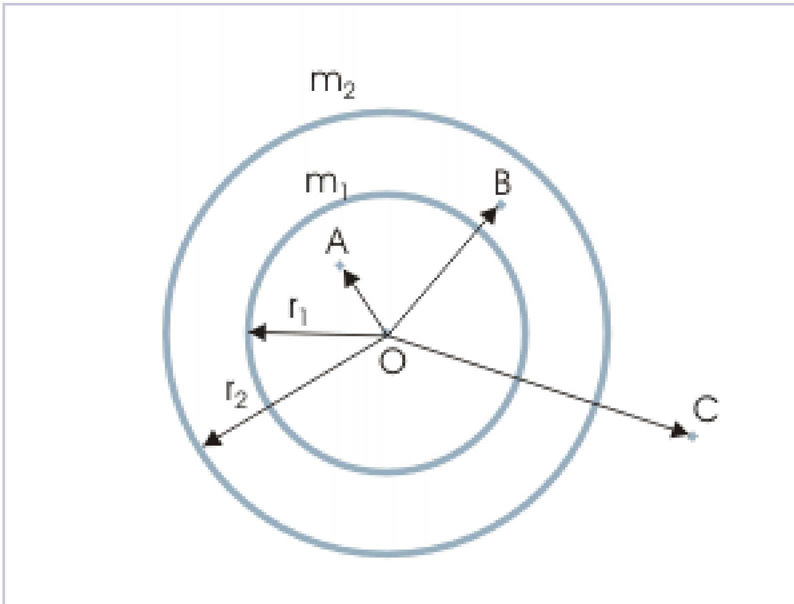

Figure 7.5: The gravitational field at three different points

The point inside smaller shell is also inside outer shell. The gravitational field inside a shell is zero. Hence, net gravitational field at a position inside the smaller shell is zero,

$$E_1 = 0$$

The gravitational field strength due to outer shell (E_o) at a point inside is zero. On the other hand, gravitational field strength due to inner shell (E_i) at a point outside is :

$$\Rightarrow E_i = \frac{GM}{r^2}$$

Hence, net gravitational field at position in between two shells is :

$$E_2 = E_i + E_o = \frac{Gm_1}{r^2}$$

A point outside outer shell is also outside inner shell. Hence, net field strength at a position outside outer shell is :

$$E_3 = Ei + Eo = \frac{Gm_1}{r^2} + \frac{Gm_2}{r^2}$$

$$\Rightarrow E_3 = \frac{G\left(m_1 + m_2\right)}{r^2}$$

Chapter 8

Gravitational potential[1]

Description of force having "action at a distance" is best described in terms of force field. The "per unit" measurement is central idea of a force field. The field strength of a gravitational field is the measure of gravitational force experienced by unit mass. On a similar footing, we can associate energy with the force field. We shall define a quantity of energy that is associated with the position of unit mass in the gravitational field. This quantity is called gravitational potential (V) and is different to potential energy as we have studied earlier. Gravitational potential energy (U) is the potential energy associated with any mass - as against unit mass in the gravitational field.

Two quantities (potential and potential energy) are though different, but are closely related. From the perspective of force field, the gravitational potential energy (U) is the energy associated with the position of a given mass in the gravitational field. Clearly, two quantities are related to each other by the equation,

$$U = mV$$

The unit of gravitational potential is Joule/kg.

There is a striking parallel among various techniques that we have so far used to study force and motion. One of the techniques employs vector analysis, whereas the other technique employs scalar analysis. In general, we study motion in terms of force (vector context), using Newton's laws of motion or in terms of energy employing "work-kinetic energy" theorem or conservation law (scalar context).

In the study of conservative force like gravitation also, we can study gravitational interactions in terms of either force (Newton's law of gravitation) or energy (gravitational potential energy). It follows, then, that study of conservative force in terms of "force field" should also have two perspectives, namely that of force and energy. Field strength presents the perspective of force (vector character of the field), whereas gravitational potential presents the perspective of energy (scalar character of field).

8.1 Gravitational potential

The definition of gravitational potential energy is extended to unit mass to define gravitational potential.

Definition 8.1: Gravitational potential

The gravitational potential at a point is equal to "negative" of the work by the gravitational force as a particle of unit mass is brought from infinity to its position in the gravitational field.

Or

Definition 8.2: Gravitational potential

The gravitational potential at a point is equal to the work by the external force as a particle of unit mass is brought from infinity to its position in the gravitational field.

[1] This content is available online at <http://cnx.org/content/m15105/1.2/>.

Mathematically,

$$V = -W_G = -\int_{\infty}^{r} \frac{F_G dr}{m} = -\int_{\infty}^{r} E dr$$

Here, we can consider gravitational field strength, "E" in place of gravitational force, " F_G " to account for the fact we are calculating work per unit mass.

8.2 Change in gravitational potential in a field due to point mass

The change in gravitational potential energy is equal to the negative of work by gravitational force as a particle is brought from one point to another in a gravitational field. Mathematically,

$$\Delta U = -\int_{r_1}^{r_2} F_G dr$$

Clearly, change in gravitational potential is equal to the negative of work by gravitational force as a particle of unit mass is brought from one point to another in a gravitational field. Mathematically, :

$$\Rightarrow \Delta V = \frac{\Delta U}{m} = -\int_{r_1}^{r_2} E dr$$

We can easily determine change in potential as a particle is moved from one point to another in a gravitational field. In order to find the change in potential difference in a gravitational field due to a point mass, we consider a point mass "M", situated at the origin of reference. Considering motion in the reference direction of "r", the change in potential between two points at a distance "r" and "r+dr" is :

Gravitational potential

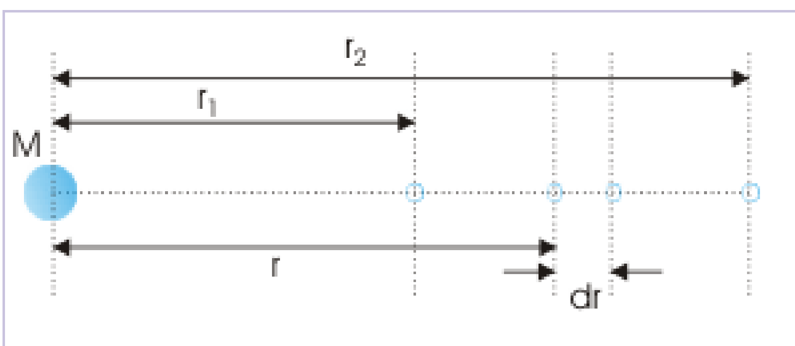

Figure 8.1: Gravitational potential difference in a gravitational field due to a point.

$$\Rightarrow \Delta V = -\int_{r_1}^{r_2} \frac{GM dr}{r^2}$$

$$\Rightarrow \Delta V = -GM \left[-\frac{1}{r}\right]_{r1}^{r2}$$

$$\Rightarrow \Delta V = GM\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

In the expression, the ratio " $\frac{1}{r_1}$ " is smaller than " $\frac{1}{r_2}$ ". Hence, change in gravitational potential is positive as we move from a point closer to the mass responsible for gravitational field to a point away from it.

8.2.1 Example

Problem 1: A particle of mass 2 kg is brought from one point to another. The increase in kinetic energy of the mass is 4 J, whereas work done by the external force is -10 J. Find potential difference between two points.

Solution : So far we have considered work by external force as equal to change in potential energy. However, if we recall, then this interpretation of work is restricted to the condition that work is done slowly in such a manner that no kinetic energy is imparted to the particle. Here, this is not the case. In general, we know from the conservation of mechanical energy that work by external force is equal to change in mechanical energy:

$$W_F = \Delta E_{mech} = \Delta K + \Delta U$$

Putting values,

$$\Rightarrow -10 = 4 + \Delta U$$

$$\Rightarrow \Delta U = -10 - 4 = -14 \quad J$$

As the change in potential energy is negative, it means that final potential energy is less than initial potential energy. It means that final potential energy is more negative than the initial.

Potential change is equal to potential energy change per unit mass. The change in potential energy per unit mass i.e. change in potential is :

$$\Rightarrow \Delta V = \frac{\Delta U}{m} = -\frac{14}{2} = -7 \quad J$$

8.3 Absolute gravitational potential in a field due to point mass

The expression for change in gravitational potential is used to find the expression for the potential at a point by putting suitable values. When,

$$V_1 = 0$$

$$V_2 = V\,(\text{say})$$

$$r_1 = \infty$$

$$r_2 = r\,(\text{say})$$

$$\Rightarrow v = -\frac{GM}{r}$$

This is the expression for determining potential at a point in the gravitational field of a particle of mass "M". We see here that gravitational potential is a negative quantity. As we move away from the particle, 1/r becomes a smaller fraction. Therefore, gravitational potential increases being a smaller negative quantity. The magnitude of potential, however, becomes smaller. The maximum value of potential is zero for r = ∞.

This relation has an important deduction. We know that particle of unit mass will move towards the particle responsible for the gravitational field, if no other force exists. This fact underlies the natural tendency of a particle to move from a higher gravitational potential (less negative) to lower gravitational potential (more negative). This deduction, though interpreted in the present context, is not specific to gravitational field, but is a general characteristic of all force fields. This aspect is more emphasized in the electromagnetic field.

8.4 Gravitational potential and field strength

A change in gravitational potential (ΔV) is equal to the negative of work by gravity on a unit mass,

$$\Delta V = -E\Delta r$$

For infinitesimal change, we can write the equation,

$$\Rightarrow dV = -Edr$$

$$\Rightarrow E = -\frac{dV}{dr}$$

Thus, if we know potential function, we can find corresponding field strength. In words, gravitational field strength is equal to the negative potential gradient of the gravitational field. We should be slightly careful here. This is a relationship between a vector and scalar quantity. We have taken the advantage by considering field in one direction only and expressed the relation in scalar form, where sign indicates the direction with respect to assumed positive reference direction. In three dimensional region, the relation is written in terms of a special vector operator called "grad".

Further, we can see here that gravitational field – a vector – is related to gravitational potential (scalar) and position in scalar form. We need to resolve this so that evaluation of the differentiation on the right yields the desired vector force. As a matter of fact, we handle this situation in a very unique way. Here, the differentiation in itself yields a vector. In three dimensions, we define an operator called "grad" as :

$$\text{grad} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)$$

where " $\frac{\partial}{\partial x}$ " is partial differentiation operator with respect to "x". This is same like normal differentiation except that it considers other dimensions (y,z) constant. In terms of "grad",

$$E = -\text{grad}\, V$$

8.5 Gravitational potential and self energy of a rigid body

Gravitational potential energy of a particle of mass "m" is related to gravitational potential of the field by the equation,

$$U = mV$$

This relation is quite handy in calculating potential energy and hence "self energy" of a system of particles or a rigid body. If we recall, then we calculated "self energy" of a system of particles by a summation process of work in which particles are brought from infinity one by one. The important point was that the gravitational force working on the particle kept increasing as more and more particles were assembled. This necessitated to calculate work by gravitational forces due to each particle present in the region, where they are assembled.

Now, we can use the "known" expressions of gravitational potential to determine gravitational potential energy of a system, including rigid body. We shall derive expressions of potential energy for few regular geometric bodies in the next module. One of the important rigid body is spherical shell, whose gravitational potential is given as :

Gravitational potential due to spherical shell

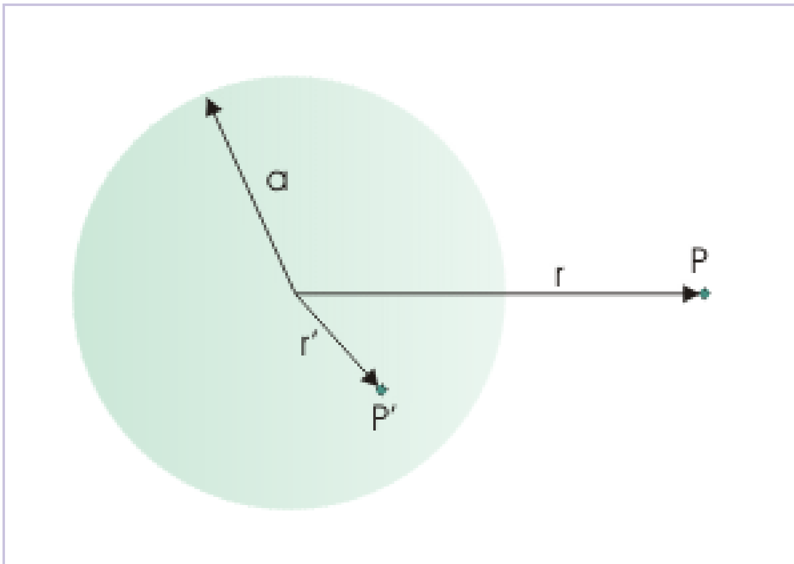

Figure 8.2: Gravitational potential at points inside and outside a spherical shell.

For a point inside or on the shell of radius "a",

$$V = -\frac{GM}{a}$$

This means that potential inside the shell is constant and is equal to potential at the surface.

For a point outside shell of radius "a" (at a linear distance, "r" from the center of shell) :

$$V = -\frac{GM}{r}$$

This means that shell behaves as a point mass for potential at a point outside the shell. These known expressions allow us to calculate gravitational potential energy of the spherical shell as explained in the section below.

8.5.1 Self energy of a spherical shell

The self potential energy is equal to work done by external force in assembling the shell bit by bit. Since zero gravitational potential energy is referred to infinity, the work needs to be calculated for a small mass at a time in bringing the same from infinity.

In order to calculate work, we draw a strategy in which we consider that some mass has already been placed symmetrically on the shell. As such, it has certain gravitational potential. When a small mass "dm" is brought, the change in potential energy is given by :

Self energy of a spherical shell

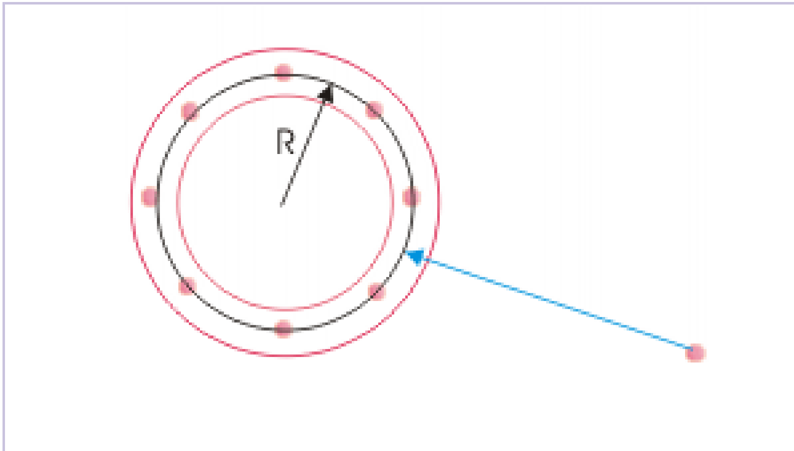

Figure 8.3: Self energy is equal to work in bringing particles one by one from the infinity.

$$dU = V dm = -\frac{Gm}{R} dm$$

We can determine total potential energy of the shell by integrating the expressions on either side of the equation,

$$\Rightarrow \int dU = -\frac{G}{R} \int m dm$$

Taking constants out from the integral on the right side and taking into account the fact that initial potential energy of the shell is zero, we have :

$$\Rightarrow U = -\frac{G}{R} \left[\frac{m^2}{2} \right]_0^M$$

$$\Rightarrow U = -\frac{GM^2}{2R}$$

This is total potential energy of the shell, which is equal to work done in bringing mass from infinity to form the shell. This expression, therefore, represents the self potential energy of the shell.

In the same manner, we can also find "self energy" of a solid sphere, if we know the expression for the gravitational potential due to a solid sphere.

Chapter 9

Gravitational potential due to rigid body[1]

We have derived expression for gravitational potential due to point mass of mass, "M", as :

$$V = -\frac{GM}{r}$$

We can find expression of potential energy for real bodies by considering the same as aggregation of small elements, which can be treated as point mass. We can, then, combine the potential algebraically to find the potential due to the body.

The derivation is lot like the derivation of gravitational field strength. There is, however, one important difference. Derivation of potential expression combines elemental potential – a scalar quantity. As such, we can add contributions from elemental parts algebraically without any consideration of direction. Indeed, it is a lot easier proposition.

Again, we are interested in finding gravitational potential due to a solid sphere, which is generally the shape of celestial bodies. As discussed earlier in the course, a solid sphere is composed of spherical shells and spherical shell, in turn, is composed of circular rings of different radii. Thus, we proceed by determining expression of potential from ring –> spherical shell –> solid sphere.

9.1 Gravitational potential due to a uniform circular ring

We need to find gravitational potential at a point "P" lying on the central axis of the ring of mass "M" and radius "a". The arrangement is shown in the figure. We consider a small mass "dm" on the circular ring. The gravitational potential due to this elemental mass is :

[1]This content is available online at <http://cnx.org/content/m15108/1.3/>.

Gravitational potential due to a uniform circular ring

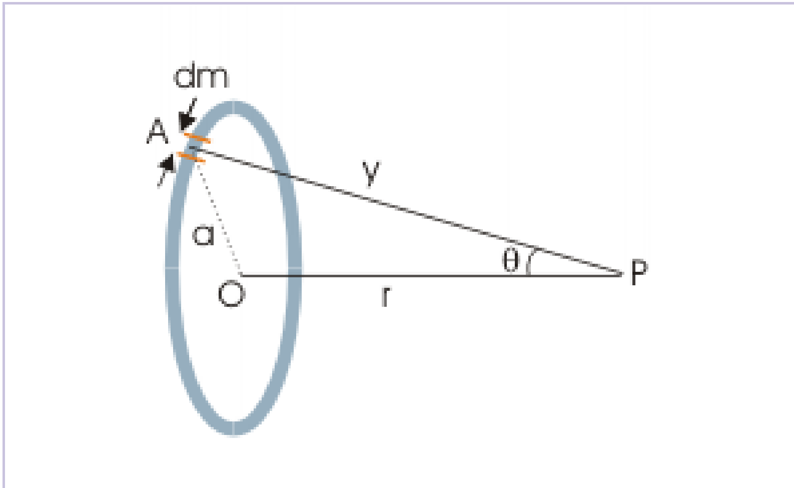

Figure 9.1: Gravitational potential at an axial point

$$dV = -\frac{Gdm}{PA} = -\frac{Gdm}{(a^2 + r^2)^{\frac{1}{2}}}$$

We can find the sum of the contribution by other elements by integrating above expression. We note that all elements on the ring are equidistant from the point, “P”. Hence, all elements of same mass will contribute equally to the potential. Taking out the constants from the integral,

$$\Rightarrow V = -\frac{G}{(a^2 + r^2)^{\frac{1}{2}}} \int_0^M dm$$

$$\Rightarrow V = -\frac{GM}{(a^2 + r^2)^{\frac{1}{2}}} = -\frac{GM}{y}$$

This is the expression of gravitational potential due to a circular ring at a point on its axis. It is clear from the scalar summation of potential due to elemental mass that the ring needs not be uniform. As no directional attribute is attached, it is not relevant whether ring is uniform or not? However, we have kept the nomenclature intact in order to correspond to the case of gravitational field, which needs to be uniform for expression as derived. The plot of gravitational potential for circular ring is shown here as we move away from the center.

Gravitational potential due to a uniform circular ring

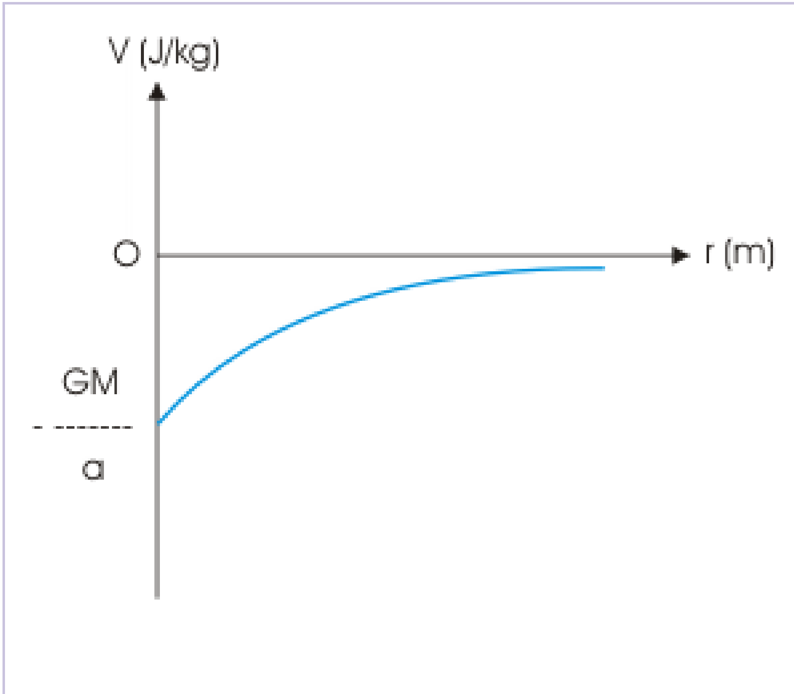

Figure 9.2: The plot of gravitational potential on axial position

Check : We can check the relationship of potential, using differential equation that relates gravitational potential and field strength.

$$\Rightarrow E = -\frac{dV}{dr} = \frac{d}{dr}\{\frac{GM}{(a^2+r^2)^{\frac{1}{2}}}\}$$

$$\Rightarrow E = GMx - \frac{1}{2}X\left(a^2+r^2\right)^{-\frac{1}{2}-1}X2r$$

$$\Rightarrow E = -\frac{GMr}{(a^2+r^2)^{\frac{3}{2}}}$$

The result is in excellent agreement with the expression derived for gravitational field strength due to a uniform circular ring.

9.2 Gravitational potential due to thin spherical shell

The spherical shell of radius "a" and mass "M" can be considered to be composed of infinite numbers of thin rings. We consider one such thin ring of infinitesimally small thickness "dx" as shown in the figure. We derive the required expression following the sequence of steps as outlined here :

Gravitational potential due to thin spherical shell

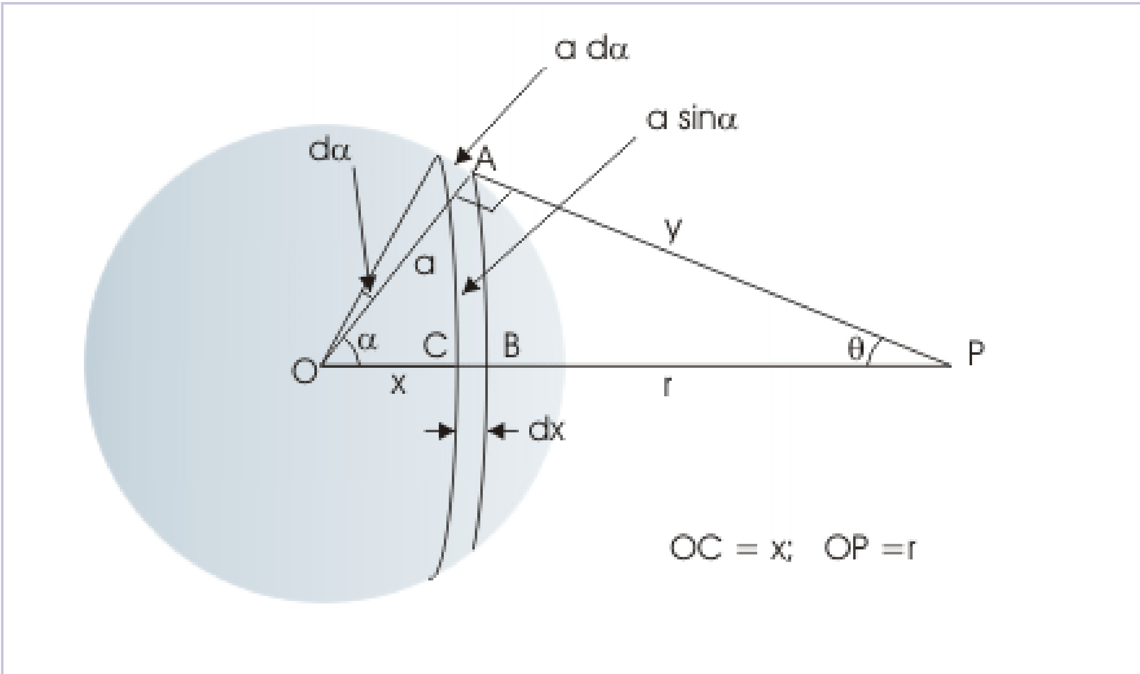

Figure 9.3: Gravitational field due to thin spherical shell at a distance "r"

(i) Determine mass of the elemental ring in terms of the mass of shell and its surface area.

$$dm = \frac{M}{4\pi a^2} X 2\pi a \sin\alpha dx = \frac{M a \sin\alpha dx}{2a^2}$$

From the figure, we see that :

$$dx = a d\alpha$$

Putting these expressions,

$$\Rightarrow dm = \frac{M a \sin\alpha dx}{2a^2} = \frac{M a \sin\alpha a d\alpha}{2a^2} = \frac{M \sin\alpha d\alpha}{2}$$

(ii) Write expression for the gravitational potential due to the elemental ring. For this, we employ the formulation derived earlier,

$$dV = -\frac{G dm}{y}$$

Putting expression for elemental mass,

$$\Rightarrow dV = -\frac{G M \sin\alpha d\alpha}{2y}$$

(i) Set up integral for the whole disc

$$\Rightarrow V = -GM \int \frac{\sin\alpha d\alpha}{2y}$$

Clearly, we need to express variables in one variable "x". From triangle, OAP,

$$y^2 = a^2 + r^2 - 2ar\cos\alpha$$

Differentiating each side of the equation,

$$\Rightarrow 2ydy = 2ar\sin\alpha d\alpha$$

$$\Rightarrow \sin\alpha d\alpha = \frac{ydy}{ar}$$

Replacing expression in the integral,

$$\Rightarrow V = -GM \int \frac{dy}{2ar}$$

We shall decide limits of integration on the basis of the position of point "P" − whether it lies inside or outside the shell. Integrating expression on right side between two general limits, initial (L_1) and final (L_2),

$$\Rightarrow V = -\frac{GM}{2ar}\left[y\right]_{L1}^{L2}$$

9.2.1 Case 1: Gravitational potential at a point outside

The total gravitational field is obtained by integrating the integral from y = r-a to y = r+a,

$$\Rightarrow V = -\frac{GM}{2ar}\left[y\right]_{r-a}^{r+a}$$

$$\Rightarrow V = -\frac{GM}{2ar}\left[\left[r + a - r + a\right]\right] = -\frac{GM}{2ar}2a$$

$$\Rightarrow V - \frac{GM}{r}$$

This is an important result. It again brings the fact that a spherical shell, for a particle outside it, behaves as if all its mass is concentrated at its center. In other words, a spherical shell can be considered as particle for an external point.

Check : We can check the relationship of potential, using differential equation that relates gravitational potential and field strength.

$$E = -\frac{dV}{dr} = \frac{d}{dr}\frac{GM}{r}$$

$$\Rightarrow E = -GMX - 1Xr^{-2}$$

$$\Rightarrow E = -\frac{GM}{r^2}$$

The result is in excellent agreement with the expression derived for gravitational field strength outside a spherical shell.

9.2.2 Case 2: Gravitational potential at a point inside

The total gravitational field is obtained by integrating the integral from y = a-r to y = a+r,

$$\Rightarrow V = -\frac{GM}{2ar}\,[y]_{a-r}^{a+r}$$

We can see here that "a-r" involves mass of the shell to the right of the point under consideration, whereas "a+r" involves mass to the left of it. Thus, total mass of the spherical shell is covered by the limits used. Now,

$$\Rightarrow V = -\frac{GM}{2ar}\,[a+r-a+r] = -\frac{GM}{2ar}2r = -\frac{GM}{a}$$

The gravitational potential is constant inside the shell and is equal to the potential at its surface. The plot of gravitational potential for spherical shell is shown here as we move away from the center.

Gravitational potential due to thin spherical shell

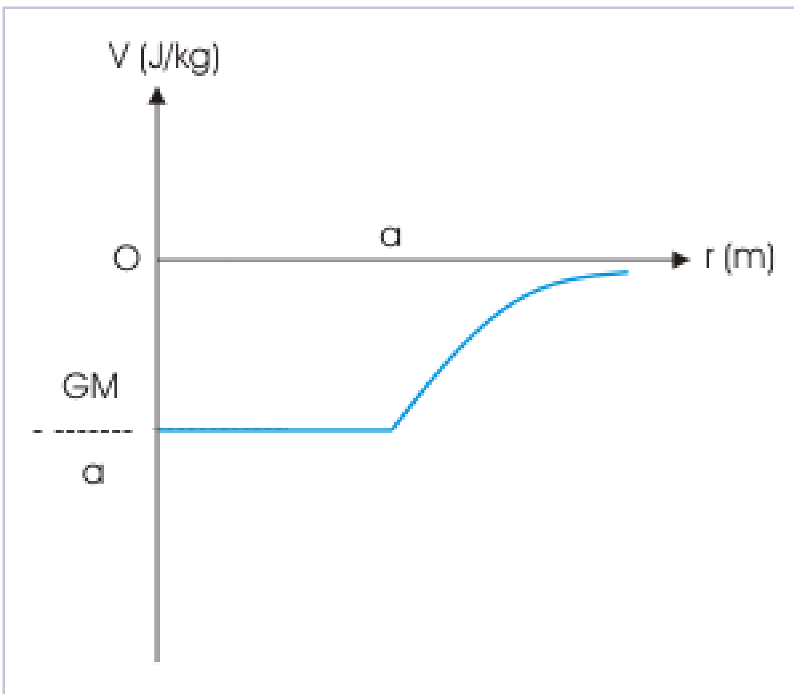

Figure 9.4: The plot of gravitational potential inside spherical shell

Check : We can check the relationship of potential, using differential equation that relates gravitational potential and field strength.

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\frac{GM}{a}$$

$$\Rightarrow E = 0$$

The result is in excellent agreement with the result obtained for gravitational field strength inside a spherical shell.

9.3 Gravitational potential due to uniform solid sphere

The uniform solid sphere of radius "a" and mass "M" can be considered to be composed of infinite numbers of thin spherical shells. We consider one such thin spherical shell of infinitesimally small thickness "dx" as shown in the figure.

Gravitational potential due to solid sphere

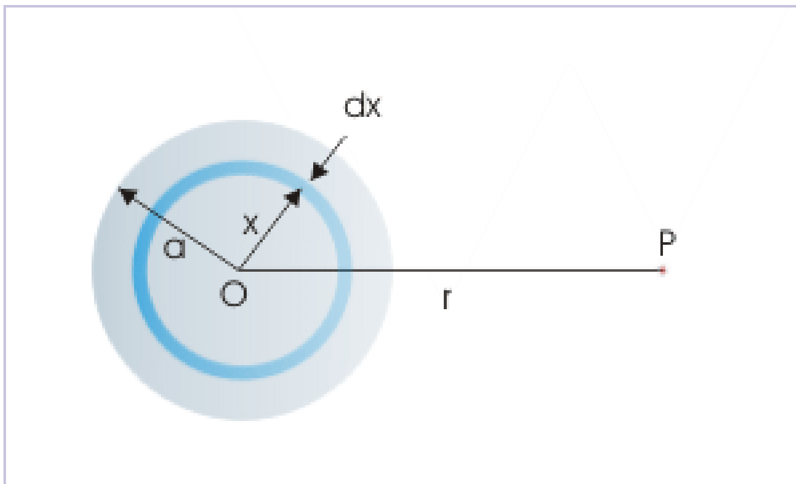

Figure 9.5: Solid sphere is composed of infinite numbers of thin spherical shells

9.3.1 Case 1 : The point lies outside the sphere

In this case, potential due to elemental spherical shell is given by :

$$dV = -\frac{Gdm}{r}$$

In this case, most striking point is that the centers of all spherical shells are coincident at center of sphere. This means that linear distance between centers of spherical shells and the point of observation is same for all shells. In turn, we conclude that the term "r" is constant for all spherical shells and as such can be taken out of the integral,

$$\Rightarrow V = -\frac{G}{r}\int dm$$

$$\Rightarrow V = -\frac{GM}{r}$$

9.3.2 Case 2 : The point lies inside the sphere

We calculate potential in two parts. For this we consider a concentric smaller sphere of radius "r" such that point "P" lies on the surface of sphere. Now, the potential due to whole sphere is split between two parts :

Gravitational potential due to solid sphere

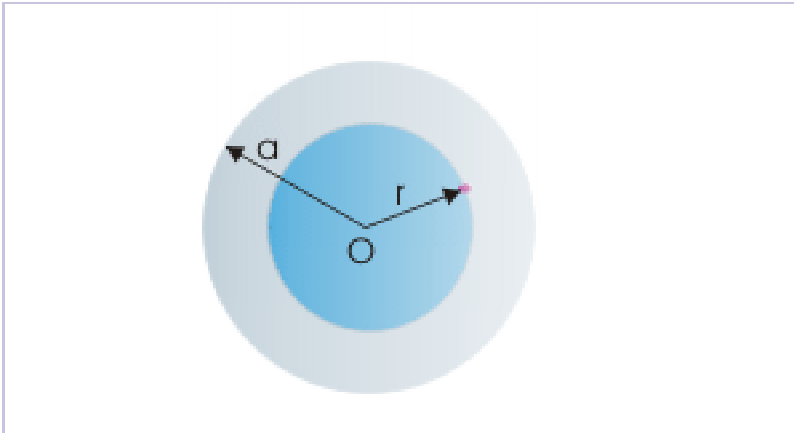

Figure 9.6: The solid sphere is split in two parts - a smaller sphere of radius "r" and remaining part of the solid sphere.

$$V = V_S + V_R$$

Where " V_S" denotes potential due to solid sphere of radius “r” and " V_S " denotes potential due to remaining part of the solid sphere between x = r and x = a. The potential due to smaller sphere is :

$$\Rightarrow V_S = -\frac{GM\prime}{r}$$

The mass, “M”’ of the smaller solid sphere is :

$$M\prime = \frac{3M}{4\pi a^3} X \frac{4\pi r^3}{3} = \frac{Mr^3}{a^3}$$

Putting in the expression of potential, we have :

$$\Rightarrow V_S = -\frac{GMr^2}{a^3}$$

In order to find the potential due to remaining part, we consider a spherical shell of thickness “dx” at a distance “x” from the center of sphere. The shell lies between x = r and x =a. The point “P” is inside this thin shell. As such potential due to the shell at point “P” inside it is constant and is equal to potential at the spherical shell. It is given by :

Gravitational potential due to remaining part

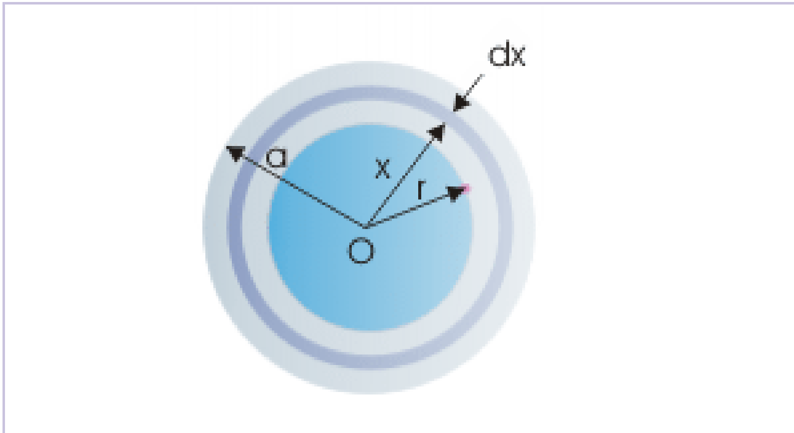

Figure 9.7: A spherical shell between point and the surface of solid sphere

$$\Rightarrow dV_R = -\frac{Gdm}{x}$$

We need to calculate the mass of the thin shell,

$$\Rightarrow M\prime = \frac{3M}{4\pi a^3} X 4\pi x^2 dx = \frac{3Mx^2 dx}{a^3}$$

Substituting in the expression of potential,

$$\Rightarrow dV_R = -\frac{G3Mx^2 dx}{a^3 x}$$

We integrate the expression for obtaining the potential at "P" between limits x = r and x =a,

$$\Rightarrow V_R = -\frac{3GM}{a^3} \int_r^a x dx$$

$$\Rightarrow V_R = -\frac{3GM}{a^3} \left[\frac{x^2}{2}\right]_r^a$$

$$\Rightarrow V_R = -\frac{3GM}{2a^3} \left(a^2 - r^2\right)$$

Adding two potentials, we get the expression of potential due to sphere at a point within it,

$$\Rightarrow V_R = -\frac{GMr^2}{a^3} - \frac{3GM}{2a^3} \left(a^2 - r^2\right)$$

$$\Rightarrow V_R = -\frac{GM}{2a^3} \left(3a^2 - r^2\right)$$

This is the expression of gravitational potential for a point inside solid sphere. The potential at the center of sphere is obtained by putting r = 0,

$$\Rightarrow V_C = -\frac{3GM}{2a}$$

This may be an unexpected result. The gravitational field strength is zero at the center of a solid sphere, but not the gravitational potential. However, it is entirely possible because gravitational field strength is rate of change in potential, which may be zero as in this case.

The plot of gravitational potential for uniform solid sphere is shown here as we move away from the center.

Gravitational potential due to solid sphere

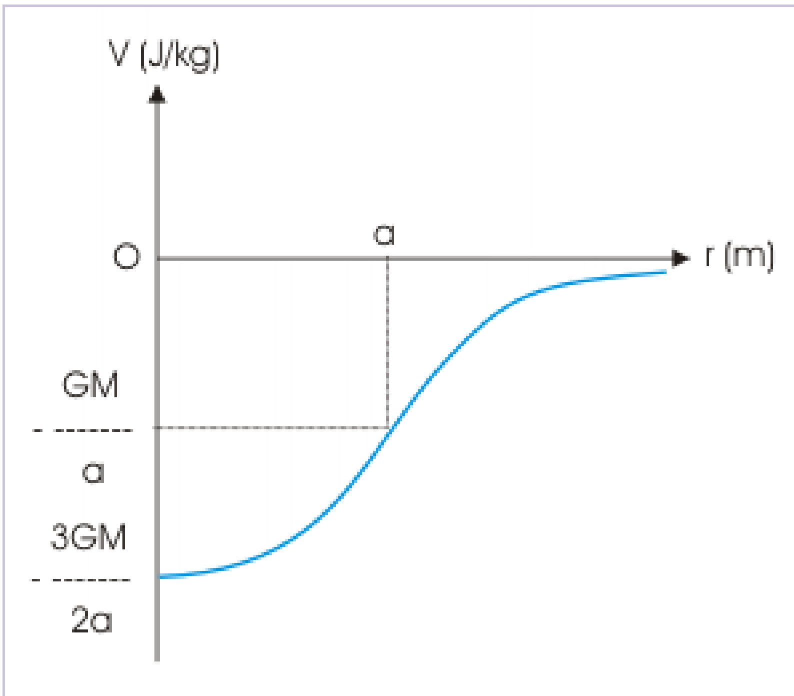

Figure 9.8: The plot of gravitational potential for uniform solid sphere

Chapter 10

Gravitational potential (application)[1]

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

10.1 Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to gravitational field. The questions are categorized in terms of the characterizing features of the subject matter :

- Potential
- Gravitational field
- Potential energy
- Conservation of mechanical energy

10.2 Potential

10.2.1

Problem 1 : A particle of mass "m" is placed at the center of a uniform spherical shell of equal mass and radius "R". Find the potential at a distance "R/4" from the center.

Solution : The potential at the point is algebraic sum of potential due to point mass at the center and spherical shell. Hence,

$$V = -\frac{Gm}{\frac{R}{4}} - \frac{Gm}{R}$$

$$\Rightarrow V = -\frac{5Gm}{R}$$

[1] This content is available online at <http://cnx.org/content/m15109/1.2/>.

10.2.2

Problem 2 : The gravitational field due to a mass distribution is given by the relation,

$$E = \frac{A}{x^2}$$

Find gravitational potential at "x".

Solution : Gravitational field is equal to negative of first differential with respect to displacement in a given direction.

$$E = -\frac{dV}{dx}$$

Substituting the given expression for "E", we have :

$$\Rightarrow \frac{A}{x^2} = -\frac{dV}{dx}$$

$$\Rightarrow dV = -\frac{Adx}{x^2}$$

Integrating between initial and final values of infinity and "x",

$$\Rightarrow \Delta V = V_f - V_i = -A \int \frac{dx}{x^2}$$

We know that potential at infinity is zero gravitational potential reference. Hence, $V_i = 0$. Let V_f = V, then:

$$\Rightarrow V = -A\,[-1/x]_{\infty}^{x} = -A\left[-\frac{1}{x} + 0\right] = \frac{A}{x}$$

10.3 Gravitational field

Problem 3 : A small hole is created on the surface of a spherical shell of mass, "M" and radius "R". A particle of small mass "m" is released a bit inside at the mouth of the shell. Describe the motion of particle, considering that this set up is in a region free of any other gravitational force.

Gravitational force

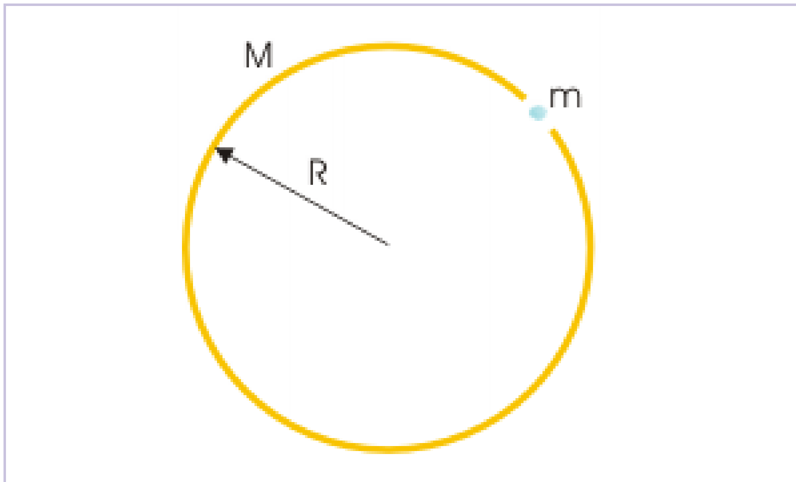

Figure 10.1: A particle of small mass "m" is released at the mouth of hole.

Solution : The gravitational potential of a shell at any point inside the shell or on the surface of shell is constant and it is given by :

$$V = -\frac{GM}{R}$$

The gravitational field,"E", is :

$$E = -\frac{dV}{dr}$$

As all quantities in the expression of potential is constant, its differentiation with displacement is zero. Hence, gravitational field is zero inside the shell :

$$E = 0$$

It means that there is no gravitational force on the particle. As such, it will stay where it was released.

10.4 Potential energy

Problem 4 : A ring of mass "M" and radius "R" is formed with non-uniform mass distribution. Find the minimum work by an external force to bring a particle of mass "m" from infinity to the center of ring.

Solution : The work done in carrying a particle slowly from infinity to a point in gravitational field is equal to potential energy of the "ring-particle" system. Now, Potential energy of the system is :

$$W_F = U = mV$$

The potential due to ring at its center is independent of mass-distribution. Recall that gravitational potential being a scalar quantity are added algebraically for individual elemental mass. It is given by :

$$V = -\frac{GM}{r}$$

Hence, required work done,

$$\Rightarrow W_F = U = -\frac{GMm}{r}$$

The negative work means that external force and displacement are opposite to each other. Actually, such is the case as the particle is attracted into gravitational field, external force is applied so that particle does not acquire kinetic energy.

10.5 Conservation of mechanical energy

Problem 5 : Imagine that a hole is drilled straight through the center of Earth of mass "M" and radius "R". Find the speed of particle of mass dropped in the hole, when it reaches the center of Earth.

Solution : Here, we apply conservation of mechanical energy to find the required speed. The initial kinetic energy of the particle is zero.

$$K_i = 0$$

On the other hand, the potential energy of the particle at the surface is :

$$Ui = mVi = -\frac{GMm}{R}$$

Let "v" be the speed of the particle at the center of Earth. Its kinetic energy is :

$$K_f = \frac{1}{2}mv^2$$

The potential energy of the particle at center of Earth is :

$$U_f = mV_f = -\frac{3GMm}{2R}$$

Applying conservation of mechanical energy,

$$K_i + U_i = K_f + U_f$$

$$\Rightarrow 0 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{3GMm}{2R}$$

$$v = \sqrt{\left(\frac{GM}{R}\right)}$$

Chapter 11

Artificial satellites[1]

The motion of a satellite or space-station is a direct consequence of Earth's gravity. Once launched in the appropriate orbit, these man-made crafts orbit around Earth without any propulsion. In this module, we shall study basics of satellite motion without going into details of the technology. Also, we shall develop analysis framework of artificial satellite, which can as well be extended to analysis of natural satellite like our moon. For the analysis here, we shall choose a simple framework of "two – body" system, one of which is Earth.

We should be aware that gravity is not the only force of gravitation working on the satellite, particularly if satellite is far off from Earth's surface. But, Earth being the closest massive body, its gravitational attraction is dominant to the extent of excluding effect of other bodies. For this reason, our analysis of satellite motion as "isolated two body system" is good first approximation.

Mass of artificial satellite is negligible in comparison to that of Earth. The "center of mass" of the "two body system" is about same as the center of Earth. There is possibility of different orbits, which are essentially elliptical with different eccentricity. A satellite close to the surface up to 2000 km describes nearly a circular trajectory. In this module, we shall confine ourselves to the analysis of satellites having circular trajectory only.

11.1 Speed of the satellite

Satellites have specific orbital speed to move around Earth, depending on its distance from the center of Earth. The satellite is launched from the surface with the help of a rocket, which parks it in particular orbit with a tangential speed appropriate for that orbit. Since satellite is orbiting along a circular path, there is requirement for the provision of centripetal force, which is always directed towards the center of orbit. This requirement of centripetal force is met by the force of gravity. Hence,

[1] This content is available online at <http://cnx.org/content/m15114/1.4/>.

Satellite

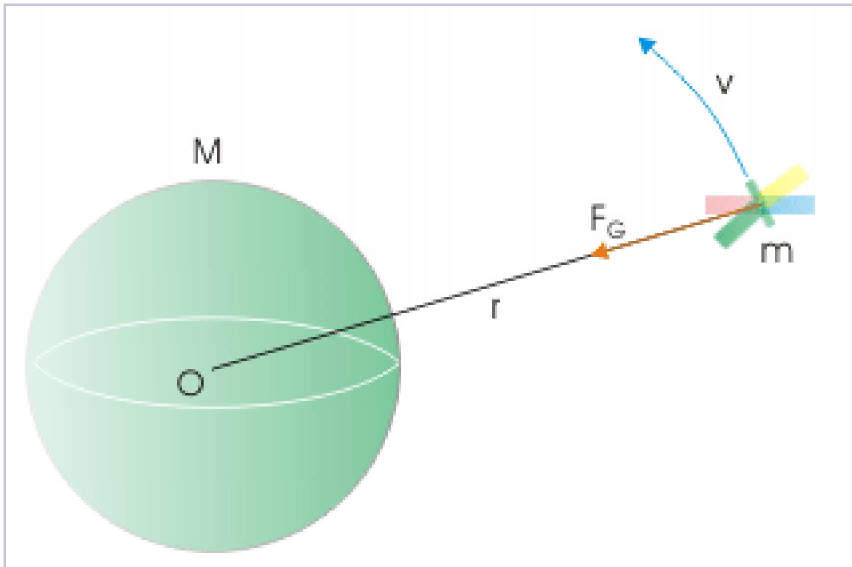

Figure 11.1: Gravitational attraction provides for the requirement of centripetal force for circular motion of satellite.

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\Rightarrow v = \sqrt{\left(\frac{GM}{r}\right)}$$

where "M" is Earth's mass and "r" is linear distance of satellite from the "center of mass" of Earth.

The important thing to realize here are : (i) orbital speed of the satellite is independent of the mass of the satellite (ii) a satellite at a greater distance moves with lesser velocity. As the product "GM" appearing in the numerator of the expression is constant, we can see that

$$\Rightarrow v \propto \frac{1}{\sqrt{r}}$$

This conclusion is intuitive in the sense that force of gravitation is lesser as we move away from Earth's surface and the corresponding centripetal force as provided by gravity is smaller. As such, orbital speed is lesser.

This fact has compounding effect on the time period of the satellite. In the first place, a satellite at a greater distance has to travel a longer distance in one revolution than the satellite closer to Earth's surface. At the same time, orbital speed is lesser as we move away. It is, then, imperative that time period of revolution increases for satellite at greater distance.

We can write the equation of orbital speed in terms of acceleration due to gravity at the surface (g = g_0), which is given by :

$$g = \frac{GM}{R^2}$$

$$\Rightarrow GM = gR^2$$

Substituting in the equation of orbital velocity, we have :

$$\Rightarrow v = \sqrt{\left(\frac{GM}{r}\right)} = \sqrt{\left(\frac{gR^2}{r}\right)} = \sqrt{\left(\frac{gR^2}{R+h}\right)}$$

where "h" is the vertical height of the satellite above the surface. Rearranging,

$$\Rightarrow v = R\sqrt{\left(\frac{g}{R+h}\right)}$$

11.2 Time period of revolution

Time period of revolution is equal to time taken to travel the perimeter of circular path. The time period of rotation is :

$$T = \frac{2\pi r}{v}$$

Substituting expression of "v" as obtained earlier,

$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{(GM)}}$$

This is the expression of time period for a satellite revolving in a circular orbit. Like orbital speed, the time period is also independent of the mass of the satellite. Now, squaring both sides, we have :

$$\Rightarrow T^2 = \frac{2\pi r^3}{GM}$$

Clearly, square of time period of a satellite is proportional the cube of the linear distance for the circular orbit,

$$\Rightarrow T^2 \propto r^3$$

11.3 Example

Problem 1: Two satellites revolve around Earth along a coplanar circular orbit in the plane of equator. They move in the same sense of direction and their periods are 6 hrs and 24 hrs respectively. The satellite having period of 6 hrs is at a distance 10000 km from the center of Earth. When the satellites are at the minimum possible separation between each other, find the magnitude of relative velocity between two satellites.

Solution : Let us denote two satellites with subscripts "1" and "2". Let the satellite designated with "1" is closer to the Earth. The positions of satellites, corresponding to minimum possible separation, are shown in the figure.

Pair of satellites

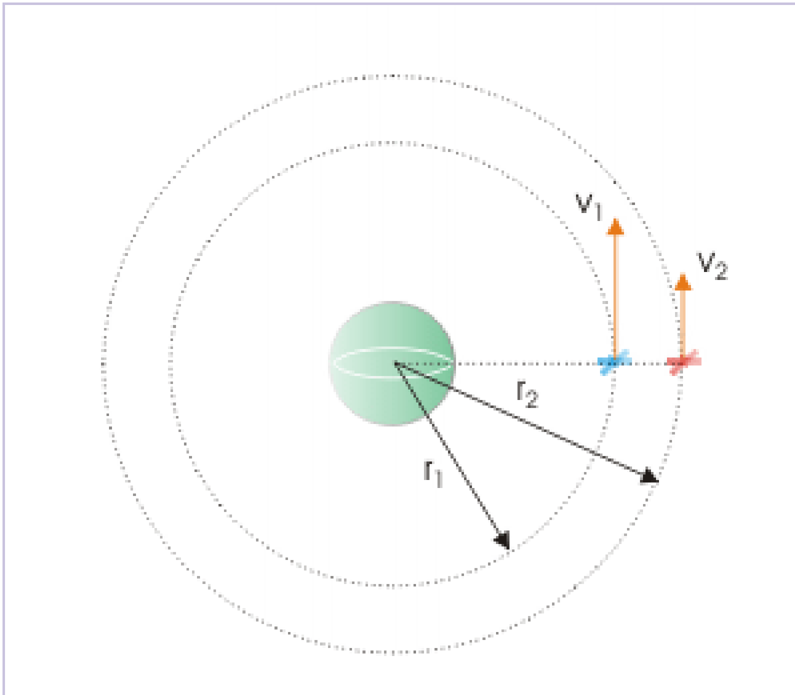

Figure 11.2: Two satellites move around Earth in two concentric circular orbits.

The distance between center of Earth and the satellite “1” is 10000 km, but this data is not available for the other satellite. However, we can evaluate other distance, using the fact that square of time period of a satellite is proportional to the cube of the linear distance for the circular orbit.

$$\frac{r_2^3}{r_1^3} = \frac{T_2^2}{T_1^2} = \left(\frac{24}{6}\right)^2 = 16$$

$$\Rightarrow \frac{r_2}{r_1} = 2$$

$$\Rightarrow r_2 = 2r_1 = 2x10^4 = 2X10^4 \quad km$$

We can now determine velocity of each satellite as :

$$v = \frac{2\pi r}{T}$$

For the first satellite,

$$\Rightarrow v_1 = \frac{2\pi X10000}{6} = \frac{10000\pi}{3}$$

For the second satellite,

$$\Rightarrow v_2 = \frac{2\pi X20000}{24} = \frac{10000\pi}{6}$$

Hence, magnitude of relative velocity is :

$$\Rightarrow v_1 - v_2 = \frac{\pi}{6} X10000 = 5238 \quad km/hr$$

11.4 Energy of "Earth-satellite" system

For consideration of energy, Earth can be treated as particle of mass "M". Thus, potential energy of "Earth – satellite" as two particles system is given by :

$$U = -\frac{GMm}{R}$$

Since expression of orbital speed of the satellite is known, we can also determine kinetic energy of the satellite as :

$$K = \frac{1}{2}mv^2$$

Putting expression of speed, "v", as determined before,

$$\Rightarrow K = \frac{GMm}{2r}$$

Note that kinetic energy of the satellite is positive, which is consistent with the fact that kinetic energy can not be negative. Now, mechanical energy is algebraic sum of potential and kinetic energy. Hence, mechanical energy of "Earth – satellite" system is :

$$\Rightarrow E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$\Rightarrow E = -\frac{GMm}{2r}$$

Here, total mechanical energy of the system is negative. We shall subsequently see that this is characteristic of a system, in which bodies are bounded together by internal force.

11.4.1 Relation among energy types

The expression of mechanical energy of the "Earth – satellite" system is typical of two body system in which one body revolves around other along a circular path. Particularly note the expression of each of the energy in the equation,

$$E = K + U$$

$$\Rightarrow -\frac{GMm}{2r} = \frac{GMm}{2r} - \frac{GMm}{r}$$

Comparing above two equations, we see that magnitude of total mechanical energy is equal to kinetic energy, but different in sign. Hence,

$$E = -K$$

Also, we note that total mechanical energy is half of potential energy. Hence,

$$E = \frac{U}{2}$$

These relations are very significant. We shall find resemblance of forms of energies in the case of Bohr's orbit as well. In that case, nucleus of hydrogen atom and electron form the two – body system and are held together by the electrostatic force.

Importantly, it provides an unique method to determine other energies, if we know any of them. For example, if the system has mechanical energy of $-200X10^6 \quad J$, then :

$$K = -E = -\left(-200X10^6\right) = 200X10^6 \quad J$$

and

$$U = 2E = -400X10^6 \quad J$$

11.4.2 Energy plots of a satellite

An inspection of the expression of energy forms reveals that that linear distance "r" is the only parameter that can be changed for a satellite of given mass, "m". From these expressions, it is also easy to realize that they have similar structure apart from having different signs. The product "GMm" is divided by "r" or "2r". This indicates that nature of variation in their values with linear distance "r" should be similar.

Energy plots

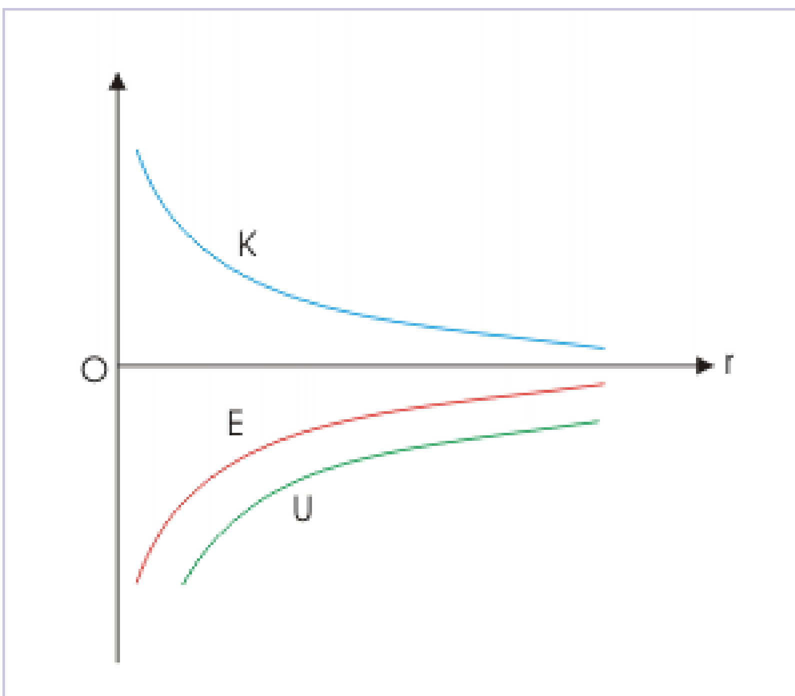

Figure 11.3: Plots of kinetic, potential and mechanical energy .vs. distance

Since kinetic energy is a positive quantity, a plot of kinetic energy .vs. linear distance, "r", is a hyperbola in the first quadrant. The expression of mechanical energy is exactly same except for the negative sign. Its plot with linear distance, therefore, is an inverted replica of kinetic energy plot in fourth quadrant. Potential energy is also negative like mechanical energy. Its plot also falls in the fourth quadrant. However, magnitude of potential energy is greater than that of mechanical energy as such the plot is displaced further away from the origin as shown in the figure.

From plots, we can conclude one important aspect of zero potential reference at infinity. From the figure, it is clear that as the distance increases and becomes large, not only potential energy, but kinetic energy also tends to become zero. We can, therefore, conclude that an object at infinity possess zero potential and kinetic energy. In other words, mechanical energy of an object at infinity is considered zero.

11.5 Gravitational binding energy

A system is bounded when constituents of the system are held together. The "Earth-satellite" system is a bounded system as members of the system are held together by gravitational attraction. Subsequently, we shall study such other bounded systems, which exist in other contexts as well. Bounded system of nucleons in a nucleus is one such example.

The characterizing aspect of a bounded system is that mechanical energy of the system is negative. However, we need to qualify that it is guaranteed to be negative when zero reference potential energy is at infinity.

Let us check out this requirement for the case of "Earth-satellite" system. The mechanical energy of Earth- satellite system is indeed negative :

$$E = -\frac{G\mathrm{Mm}}{2r}$$

where "M" and "m" are the mass of Earth and satellite. Hence, "Earth - satellite" system is a bounded system.

We can infer from the discussion of a bounded system that the "binding energy" is the amount of energy required to disintegrate (dismember) a bounded system. For example, we can consider a pebble lying on Earth's surface. What is the energy required to take this pebble far off in the interstellar space, where Earth's gravity ceases to exist? We have seen that infinity serves as a theoretical reference, where gravitational field ceases to exits. Further, if we recall, then potential energy is defined as the amount of work done by external agency to bring a particle slowly from infinity to a position in gravitational field. The work by external force is negative as its acts opposite to the displacement. Clearly, taking pebble to the infinity is reverse action. Work by external force is in the direction of displacement. As such, work done in this case is positive. Therefore, binding energy of the pebble is a positive quantity and is equal to the magnitude of potential energy for the pebble. If its mass is "m", then binding energy of the "Earth-pebble" system is :

$$\Rightarrow E_B = -U = -\left(-\frac{G\mathrm{Mm}}{r}\right) = \frac{G\mathrm{Mm}}{r}$$

where "M" and "m" are the mass of Earth and pebble respectively and "R" is the radius of Earth.

This is, however, a specific description of dismembering process. In general, a member of the system will have kinetic energy due to its motion. Let us consider the case of "Earth-satellite" system. The satellite has certain kinetic energy. If we want to take this satellite to infinity, we would first require to bring the satellite to a dead stop and then take the same to infinity. Therefore, binding energy of the system is a positive quantity, which is equal to the magnitude of the mechanical energy of the system.

Definition 11.1: Binding energy

Binding energy is equal to the modulus of mechanical energy.

Going by the definition, the binding energy of the "Earth-satellite" system is :

$$\Rightarrow E_B = -E = -\left(-\frac{G\mathrm{Mm}}{2r}\right) = \frac{G\mathrm{Mm}}{2r}$$

where "r" is the linear distance between the center of Earth and satellite.

11.6 Satellite systems

The satellites are made to specific tasks. One of the most significant applications of artificial satellite is its use in telecast around the world. Earlier it was difficult to relay telecast signals due to spherical shape of Earth. In recent time, advancements in communication have brought about astounding change in the way we live. The backbone of this communication wonder is variety of satellite systems orbiting around Earth.

Satellite systems are classified for different aspects of satellite motion. From the point of physics, it is the orbital classification of satellite systems, which is more interesting. Few of the famous orbits are described here. Almost all orbits generally describe an elliptical orbit. We shall discuss elliptical orbits in the module dedicated to Kepler's law. For the present, however, we can approximate them to be circular for analysis purpose.

1: Geocentric orbit : It is an orbit around Earth. This is the orbit of artificial satellite, which is launched to revolve around Earth. Geocentric orbit is further classified on the basis of distance from Earth's surface (i) low Earth orbit up to 2000 km (ii) middle Earth orbit between 2000 and geo-synchronous orbit (36000 km) and (iii) high Earth orbit above geo-synchronous orbit (36000 km).

2: Heliocentric Orbit : It is an orbit around Sun. The orbits of planets and all other celestial bodies in the solar system describe heliocentric orbits.

3: Geosynchronous Orbit : The time period of this orbit is same as the time period of Earth.

4: Geostationary Orbit : The plane of rotation is equatorial plane. The satellite in this orbit has time period equal to that of Earth. Thus, motion of satellite is completely synchronized with the motion of Earth. The sense of rotation of the satellite is same as that of Earth. The satellite, therefore, is always above a given position on the surface. The orbit is at a distance of 36000 km from Earth's surface and about 42400 (= 36000 + 6400) km from the center of Earth. The orbit is also known as Clarke's orbit after the name of author, who suggested this orbit.

5: Molniya Orbit – It is an orbit having inclination of $63.4\,^\circ$ with respect to equatorial plane and orbital period equal to half that of Earth.

6: Polar orbit : The orbit has an inclination of $90\,^\circ$ with respect to the equatorial plane and as such, passes over Earth's poles.

Another important classification of satellite runs along the uses of satellites. Few important satellite types under this classification are :

1: Communication satellites : They facilitate communication around the world. The geostationary satellite covers ground locations, which are close to equator. Geostationary satellites appears low from a positions away from equator. For locations at different latitudes away from equator, we need to have suitably designed orbits so that the area can be covered round the clock. Molniya orbit is one such orbit, which is designed to provide satellite coverage through a satellite system, consisting of more than one satellite.

2: Astronomical satellites : They are designed for studying celestial bodies.

3: Navigational satellites : They are used to specify location on Earth and develop services based on navigation.

4: Earth observation satellites : They are designed for studying Earth system, environment and disaster management.

5: Weather satellites : They facilitate to monitor weather and related services.

6: Space station : It is an artificial structure in space for human beings to stay and do assigned experiments/works

As a matter of fact, there is quite an elaborate classification system. We have only named few important satellite systems. In particular, there are varieties of satellite systems, including reconnaissance satellites, to meet military requirement.

11.7 Acknowledgment

Author wishes to thank Arunabha guha, Physics dept, Georgian court university, Lakewood, New jersey, USA for pointing out a mistake in the example contained in this module.

Chapter 12

Projection in gravitational field[1]

Gravitational force of attraction is a binding force. An object requires certain minimum velocity to break free from this attraction. We are required to impart object with certain kinetic energy to enable it to overcome gravitational pull. As the object moves away, gravitational pull becomes smaller. However, at the same time, speed of the object gets reduced as kinetic energy of the object is continuously transferred into potential energy. Remember, potential energy is maximum at the infinity.

Depending on the initial kinetic energy imparted to the projectile, it will either return to the surface or will move out of the Earth's gravitational field.

The motion of a projectile, away from Earth's surface, is subjected to variable force – not a constant gravity as is the case for motion near Earth's surface. Equivalently, acceleration due to gravity, "g", is no more constant at distances thousands of kilometers away. As such, equations of motion that we developed and used (like v = u+at) for constant acceleration is not valid for motion away from Earth.

We have already seen that analysis using energy concept is suitable for such situation, when acceleration is not constant. We shall, therefore, develop analysis technique based on conservation of energy.

12.1 Context of motion

We need to deal with two forces for projectile : air resistance i.e. friction and gravitational force. Air resistance is an external non-conservative force, whereas gravity is an internal conservative force to the "Earth-projectile" system. The energy equation for this set up is :

$$W_F = \Delta K + \Delta U$$

Our treatment in the module, however, will neglect air resistance for mathematical derivation. This is a base consideration for understanding motion of an object in a gravitational field at greater distances. Actual motion will not be same as air resistance at higher velocity generates tremendous heat and the projectile, as a matter of fact, will either burn up or will not reach the distances as predicted by the analysis. Hence, we should keep this limitation of our analysis in mind.

Nevertheless, the situation without friction is an ideal situation to apply law of conservation of energy. There is only conservative force in operation on the object in translation. The immediate consequence is that work by this force is independent of path. As there is no external force on the system, the changes takes place between potential and kinetic energy in such a manner that overall change in mechanical energy always remains zero. In other words, only transfer of energy between kinetic and gravitational potential energy takes place. As such,

$$\Rightarrow \Delta K + \Delta U = 0$$

[1] This content is available online at <http://cnx.org/content/m15150/1.3/>.

12.1.1 Change in potential energy

Earlier, we used the expression "mgh" to compute potential energy or change in potential energy. We need to correct this formula for determining change in potential energy by referring calculation of potential energy to infinity. Using formula of potential energy with infinity as reference, we determine the potential difference between Earth's surface and a point above it, as :

Gravitational potential difference

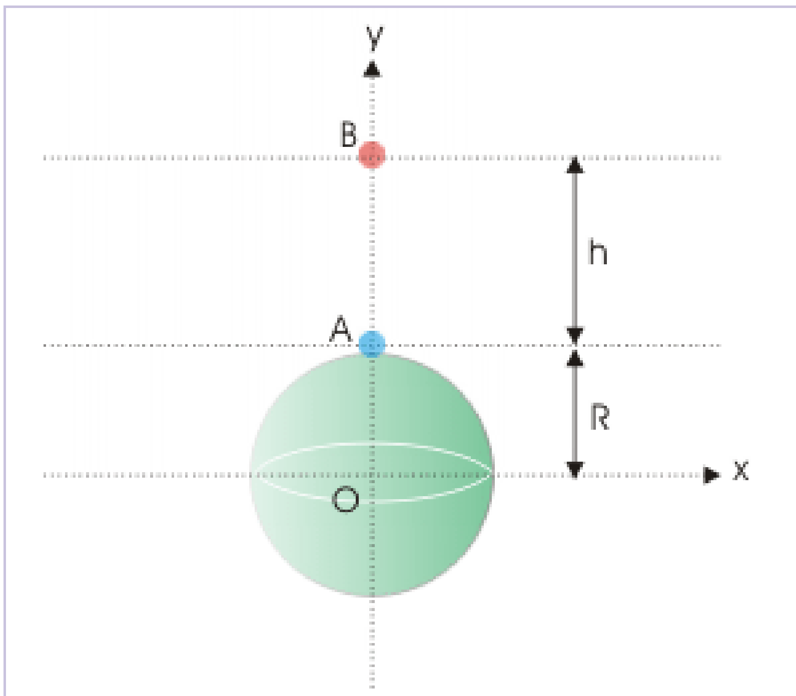

Figure 12.1: An object at a height "h"

$$\Rightarrow \Delta U = -\frac{GMm}{(R+h)} - \left(-\frac{GMm}{R}\right)$$

$$\Rightarrow \Delta U = GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

We can eliminate reference to gravitational constant and mass of Earth by using relation of gravitational acceleration at Earth's surface ($g = g_0$),

$$g = \frac{GM}{R^2}$$

$$\Rightarrow GM = gR^2$$

Substituting in the equation of change in potential energy, we have :

$$\Rightarrow \Delta U = mgR^2\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$\Rightarrow \Delta U = mgR^2 X \frac{h}{R\,(R+h)}$$

$$\Rightarrow \Delta U = \frac{mgh}{1+\frac{h}{R}}$$

It is expected that this general formulation for the change in potential energy should be reduced to approximate form. For h≪R, we can neglect "h/R" term and,

$$\Rightarrow \Delta U = mgh$$

12.1.2 Maximum Height

For velocity less than escape velocity (the velocity at which projectile escapes the gravitation field of Earth), the projected particle reaches a maximum height and then returns to the surface of Earth.

When we consider that acceleration due to gravity is constant near Earth's surface, then applying conservation of mechanical energy yields :

$$\frac{1}{2}mv^2 + 0 = 0 + mgh$$

$$\Rightarrow h = \frac{v^2}{2g}$$

However, we have seen that "mgh" is not true measure of change in potential energy. Like in the case of change in potential energy, we come around the problem of variable acceleration by applying conservation of mechanical energy with reference to infinity.

Maximum height

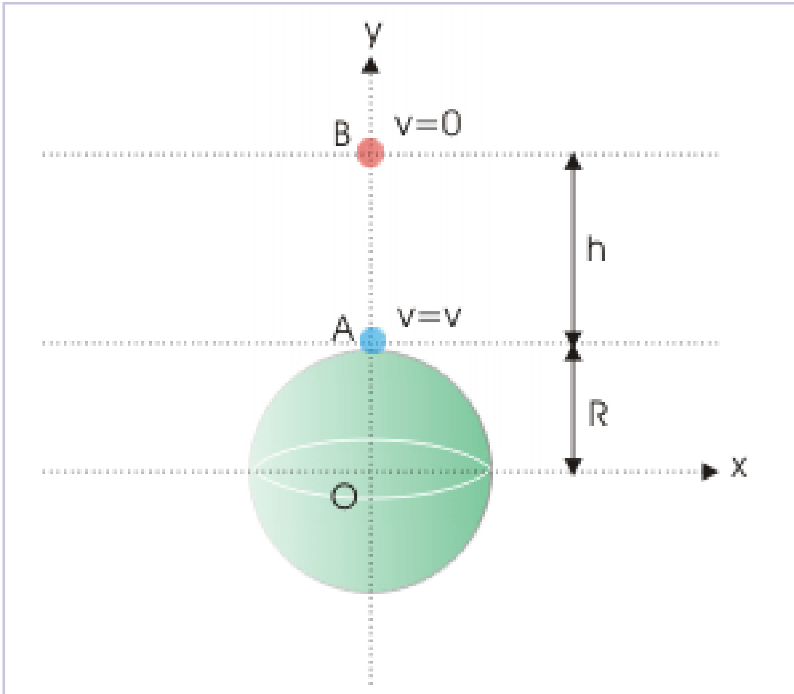

Figure 12.2: The velocity is zero at maximum height, "h".

$$K_i + U_i = K_f + U_f$$

$$\Rightarrow -\frac{GMm}{R} + \frac{1}{2}mv^2 = 0 + -\frac{GMm}{R+h}$$

$$\Rightarrow \frac{GM}{R+h} = \frac{GM}{R} - \frac{v^2}{2}$$

$$\Rightarrow R+h = \frac{GM}{\frac{GM}{R} - \frac{v^2}{2}}$$

$$\Rightarrow h = \frac{GM}{\frac{GM}{R} - \frac{v^2}{2}} - R$$

$$\Rightarrow h = \frac{GM - GM + \frac{v^2 R}{2}}{\frac{GM}{R} - \frac{v^2}{2}}$$

$$\Rightarrow h = \frac{v^2 R}{\frac{2GM}{R} - v^2}$$

We can also write the expression of maximum height in terms of acceleration at Earth's surface using the relation :

$$\Rightarrow GM = gR^2$$

Substituting in the equation and rearranging,

$$\Rightarrow h = \frac{v^2}{2g - \frac{v^2}{R}}$$

This is the maximum height attained by a projection, which is thrown up from the surface of Earth.

12.1.3 Example

Problem 1: A particle is projected vertically at 5 km/s from the surface Earth. Find the maximum height attained by the particle. Given, radius of Earth = 6400 km and g = 10 m/s^2.

Solution : We note here that velocity of projectile is less than escape velocity 11.2 km/s. The maximum height attained by the particle is given by:

$$h = \frac{v^2}{2g - \frac{v^2}{R}}$$

Putting values,

$$\Rightarrow h = \frac{\left(5X10^3\right)^2}{2X10 - \frac{(5X10^3)^2}{(6.4X10^6)}}$$

$$\Rightarrow h = \frac{\left(25X10^6\right)}{2X10 - \frac{(25X10^6)}{(6.4X10^6)}}$$

$$\Rightarrow h = 1.55x10^6 = 1550000 \quad m = 1550 \quad km$$

It would be interesting to compare the result, if we consider acceleration to be constant. The height attained is :

$$h = \frac{v^2}{2}g = 25X\frac{10^6}{20} = 1.25X10^6 = 1250000 \quad m = 1250 \quad km$$

As we can see, approximation of constant acceleration due to gravity, results in huge discrepancy in the result.

12.2 Escape velocity

In general, when a body is projected up, it returns to Earth after achieving a certain height. The height of the vertical flight depends on the speed of projection. Greater the initial velocity greater is the height attained.

Here, we seek to know the velocity of projection for which body does not return to Earth. In other words, the body escapes the gravitational influence of Earth and moves into interstellar space. We can know this velocity in verities of ways. The methods are equivalent, but intuitive in approach. Hence, we shall present here all such considerations :

12.2.1 1: Binding energy :

Gravitational binding energy represents the energy required to eject a body out of the influence of a gravitational field. It is equal to the energy of the system, but opposite in sign. In the absence of friction, this energy is the mechanical energy (sum of potential and kinetic energy) in gravitational field.

Now, it is clear from the definition of binding energy itself that the initial kinetic energy of the projection should be equal to the binding energy of the body in order that it moves out of the gravitational influence.

Now, the body to escape is at rest before being initiated in projection. Thus, its binding energy is equal to potential energy only.

Therefore, kinetic energy of the projection should be equal to the magnitude of potential energy on the surface of Earth,

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

where " v_e " is the escape velocity. Note that we have used "R" to denote Earth's radius, which is the distance between the center of Earth and projectile on the surface. Solving above equation for escape velocity, we have :

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)}$$

12.2.2 2. Conservation of mechanical energy :

The act of putting a body into interstellar space is equivalent to taking the body to infinity i.e. at a very large distance. Infinity, as we know, has been used as zero potential energy reference. The reference is also said to represent zero kinetic energy.

From conservation of mechanical energy, it follows that total mechanical energy on Earth's should be equal to mechanical energy at infinity i.e. should be equal to zero. But, we know that potential energy at the surface is given by :

$$U = -\frac{GMm}{R}$$

On the other hand, for body to escape gravitational field,

$$K + U = 0$$

Therefore, kinetic energy required by the projectile to escape is :

$$\Rightarrow K = -U = \frac{GMm}{R}$$

Now, putting expression for kinetic energy and proceeding as in the earlier derivation :

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)}$$

12.2.3 3: Final velocity is not zero :

We again use conservation of mechanical energy, but with a difference. Let us consider that projected body of mass, "m" has initial velocity "u" and an intermediate velocity, "v", at a height "h". The idea here is to find condition for which intermediate velocity ,"v", never becomes zero and hence escape Earth's influence. Applying conservation of mechanical energy, we have :

$$E_i = E_f$$

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mu^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{GMm}{R+h}$$

Rearranging,

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{GMm}{R} + \frac{GMm}{R+h}$$

In order that, final velocity ("v") is positive, the expressions on the right should evaluate to a positive value. For this,

$$\Rightarrow \frac{1}{2}mu^2 \geq \frac{GMm}{R}$$

For the limiting case, u $= v_e$,

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)}$$

12.3 Interpreting escape velocity

These three approaches to determine escape velocity illustrates how we can analyze a given motion in gravitational field in many different ways. We should be aware that we have determined the minimum velocity required to escape Earth's gravity. It is so because we have used the expression of potential energy, which is defined for work by external force slowly.

However, it is found that the velocity so calculated is good enough for escaping gravitational field. Once projected body achieves considerable height, the gravitational attraction due to other celestial bodies also facilitates escape from Earth's gravity.

Further, we can write the expression of escape velocity in terms of gravitational acceleration (consider $g = g_0$),

$$g = \frac{GM}{r^2}$$

$$\Rightarrow \frac{GM}{r} = gr$$

Putting in the expression of escape velocity, we have :

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)} = \sqrt{(2gR)}$$

12.3.1 Escape velocity of Earth

In the case of Earth,

$$M = 5.98X10^{24} \quad kg$$

$$R = 6.37X10^{6} \quad m$$

$$\Rightarrow v_e = \sqrt{\left(\frac{2X6.67X10^{-11}X5.98X10^{24}}{6.37X10^{6}}\right)}$$

$$\Rightarrow v_e = 11.2 \quad km/s$$

We should understand that this small numerical value is deceptive. Actually, it is almost impossible to impart such magnitude of speed. Let us have a look at the magnitude in terms of "km/hr",

$$v_e = 11.2 \quad km/s = 11.2X60x60 = 40320 \quad km/hr$$

If we compare this value with the speed of modern jet (which moves at 1000 km/hr), this value is nearly 40 times! It would be generally a good idea to project object from an artificial satellite instead, which itself may move at great speed of the order of about 8-9 km/s. The projectile would need only additional 2 or 3 km/hr of speed to escape, if projected in the tangential direction of the motion of the satellite.

This mechanism is actually in operation in multistage rockets. Each stage acquires the speed of previous stage. The object (probe or vehicle) can, then, be let move on its own in the final stage to escape Earth's gravity. The mechanism as outlined here is actually the manner an interstellar probe or vehicle is sent out of the Earth's gravitational field. We can also appreciate that projection, in this manner, has better chance to negotiate friction effectively as air resistance at higher altitudes is significantly less or almost negligible.

This discussion of escape velocity also underlines that the concept of escape velocity is related to object, which is not propelled by any mechanical device. An object, if propelled, can escape gravitational field at any speed.

Escape velocity of Moon :

In the case of Earth's moon,

$$M = 7.4X10^{20} \quad kg$$

$$R = 1.74X10^{6} \quad m$$

$$\Rightarrow v_e = \sqrt{\left(\frac{2X6.67X10^{-11}X7.4X10^{20}}{1.74X10^{6}}\right)}$$

$$\Rightarrow v_e = 2.4 \quad km/s$$

The root mean square velocity of gas is greater than this value. This is the reason, our moon has no atmosphere. Since sound requires a medium to propagate, we are unable to talk directly there as a consequence of the absence of atmosphere.

12.3.2 Direction of projection

It may appear that we may need to fire projectile vertically to let it escape in interstellar space. This is not so. The spherical symmetry of Earth indicates that we can project body in any direction with the velocity as determined such that it clears physical obstructions in its path. From this point of view, the term "velocity" is a misnomer as direction of motion is not involved. It would have been more appropriate to call it "speed".

The direction, however, makes a difference in escape velocity for some other reason. The Earth rotates in particular direction – it rotates from East to west at a linear speed of 465 m/s. So if we project the body in the tangential direction east-ward, then Earth rotation helps body's escape. The effective escape velocity is 11200 – 465 = 10735 m/s. On the other hand, if we project west-ward, then escape velocity is 11200 + 465 = 11635 m/s.

12.3.3 Escape velocity and Black hole

Black holes are extremely high density mass. This represents the final stage of evolution of a massive star, which collapses due to its own gravitational force. Since mass remains to be very large while radius is reduced (in few kms), the gravitational force becomes extremely large. Such great is the gravitational force that it does not even allow light to escape.

Lesser massive star becomes neutron star instead of black hole. Even neutron star has very high gravitational field. We can realize this by calculating escape velocity for one such neutron star,

$$M = 3X10^{30} \quad kg$$

$$R = 3X10^{4} \quad m$$

$$\Rightarrow v_e = \sqrt{\left(\frac{2X6.67X10^{-11}X3X10^{30}}{3X10^{4}}\right)}$$

$$\Rightarrow v_e = 1.1X10^{5} \quad m/s$$

It is quite a speed comparable with that of light. Interstellar Black hole is suggested to be 5 times the mass of neutron star and 10 times the mass of sun! On the other hand, its dimension is in few kilometers. For this reason, following is possible :

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)} > c$$

where "c" is the speed of light. Hence even light will not escape the gravitational force of a black hole as the required velocity for escape is greater than speed of light.

12.4 Nature of trajectory

In this section, we shall attempt to analyze trajectory of a projectile for different speed range. We shall strive to get the qualitative assessment of the trajectory – not a quantitative one.

In order to have a clear picture of the trajectory of a projectile, let us assume that a projectile is projected from a height, in x-direction direction as shown. The point of projection is, though, close to the surface; but for visualization, we have shown the same at considerable distance in terms of the dimension of Earth.

Projection in Earth's gravitation

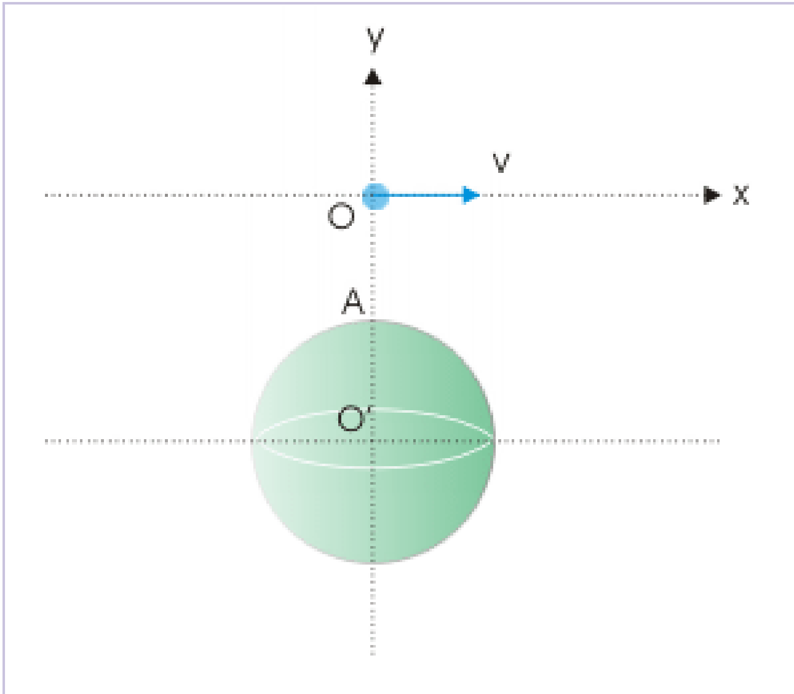

Figure 12.3: Projectile is projected with certain velocity in x-direction.

Let " v_O " be the speed of a satellite near Earth's surface and " v_e " be the escape velocity for Earth's gravity. Then,

$$v_O = \sqrt{(gR)}$$

$$v_e = \sqrt{(2gR)}$$

Different possibilities are as following :

1 : $v = 0$: The gravity pulls the projectile back on the surface. The trajectory is a straight line (OA shown in the figure below).

2 : $v < v_C$: We denote a projection velocity " v_C " of the projectile such that it always clears Earth's surface (OC shown in the figure below). A limiting trajectory will just clear Earth's surface. If the projection velocity is less than this value then the trajectory of the projectile will intersect Earth and projectile will hit the surface (OB shown in the figure below).

Projection in Earth's gravitation

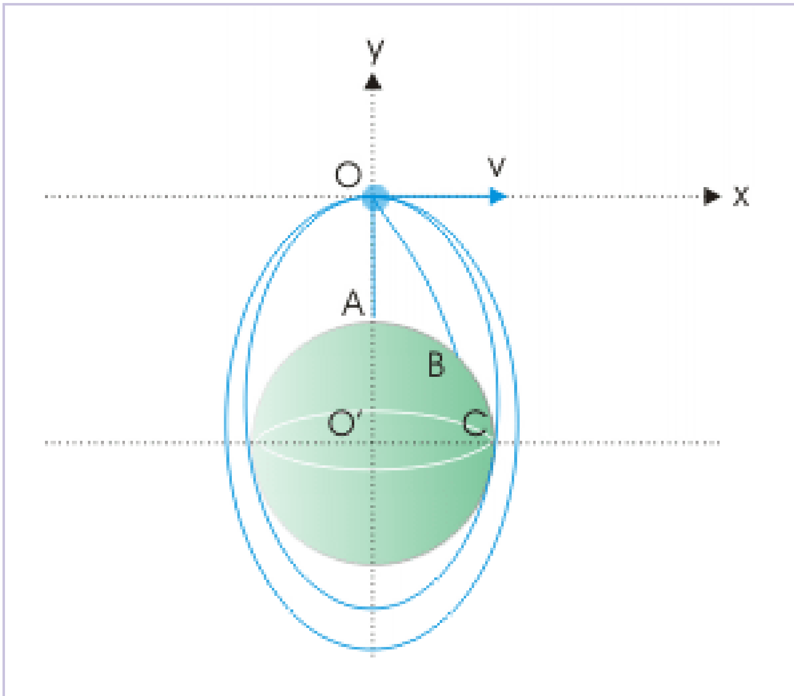

Figure 12.4: Projectile is projected with certain velocity in x-direction.

3 : $v_C < v < v_O$: Since projection velocity is greater than limiting velocity to clear Earth and less than the benchmark velocity of a satellite in circular orbit, the projectile will move along an elliptical orbit. The Earth will be at one of the foci of the elliptical trajectory (see figure above).

4 : $v = v_O$: The projectile will move along a circular trajectory (see inner circle in the figure below).

Projection in Earth's gravitation

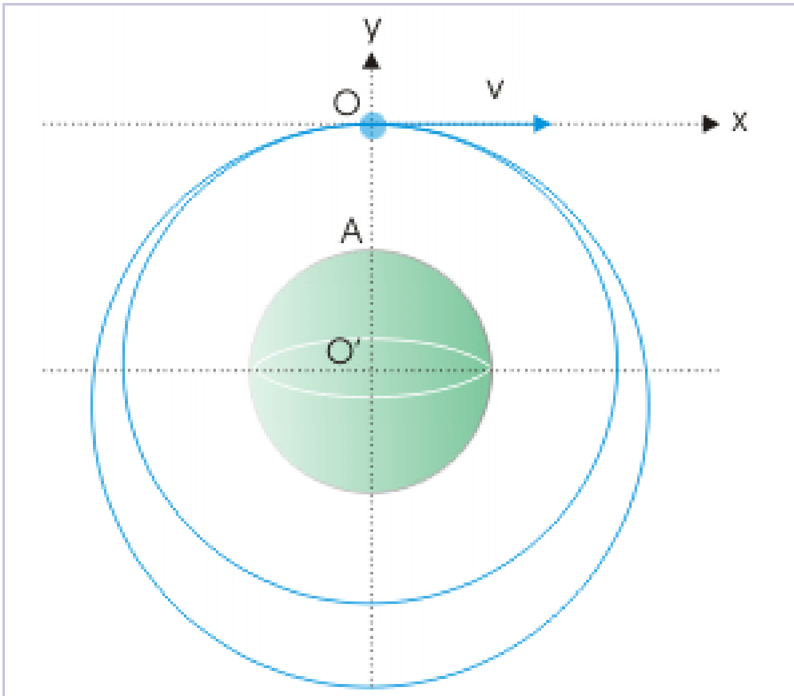

Figure 12.5: Projectile is projected with certain velocity in x-direction.

5 : $v_O < v < v_e$: The projection velocity is greater than orbital velocity for circular trajectory, the path of the projectile is not circular. On the other hand, since projection velocity is less than escape velocity, the projectile will not escape gravity either. It means that projectile will be bounded to the Earth. Hence, trajectory of the projectile is again elliptical with Earth at one of the foci (see outer ellipse in the figure above).

6 : $v = v_e$: The projectile will escape gravity. In order to understand the nature of trajectory, we can think of force acting on the particle and resulting motion. The gravity pulls the projectile in the radial direction towards the center of Earth. Thus, projectile will have acceleration in radial direction all the time. The component of gravity along x-direction is opposite to the direction of horizontal component of velocity. As such, the particle will be retarded in x-direction. On the other hand, vertical component of gravity will accelerate projectile in the negative y – direction.

Projection in Earth's gravitation

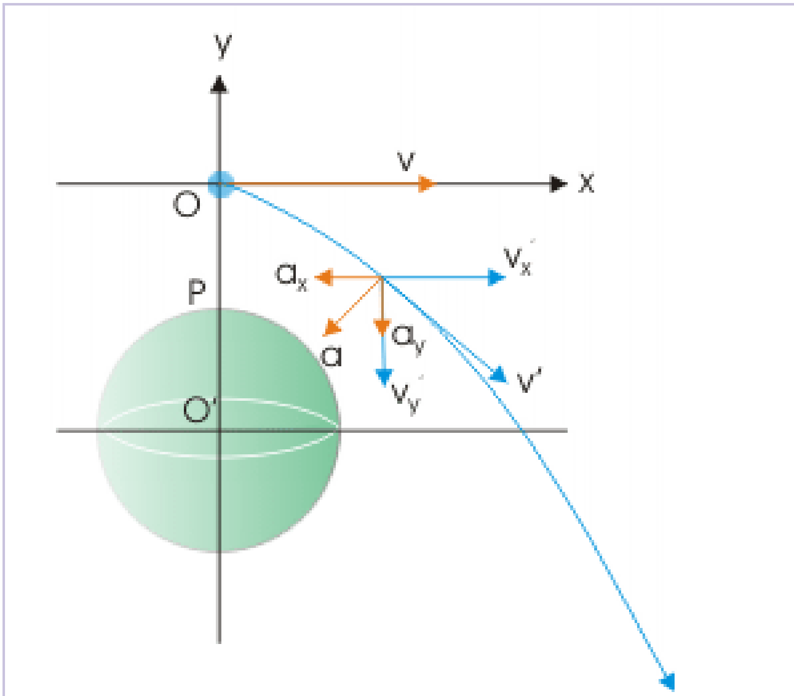

Figure 12.6: Projectile is projected with certain velocity in x-direction.

However, as the projection speed of the projectile is equal to escape velocity, the projectile will neither be intersected by Earth's surface nor be bounded to the Earth. The resulting trajectory is parabola leading to the infinity. It is an open trajectory.

7 : $v > v_e$: We can infer that projection velocity is just too great. The impact of gravity will be for a very short duration till the projectile is close to Earth. However, as distance increases quickly, the impact of gravitational force becomes almost negligible. The final path is parallel to x-direction.

Projection in Earth's gravitation

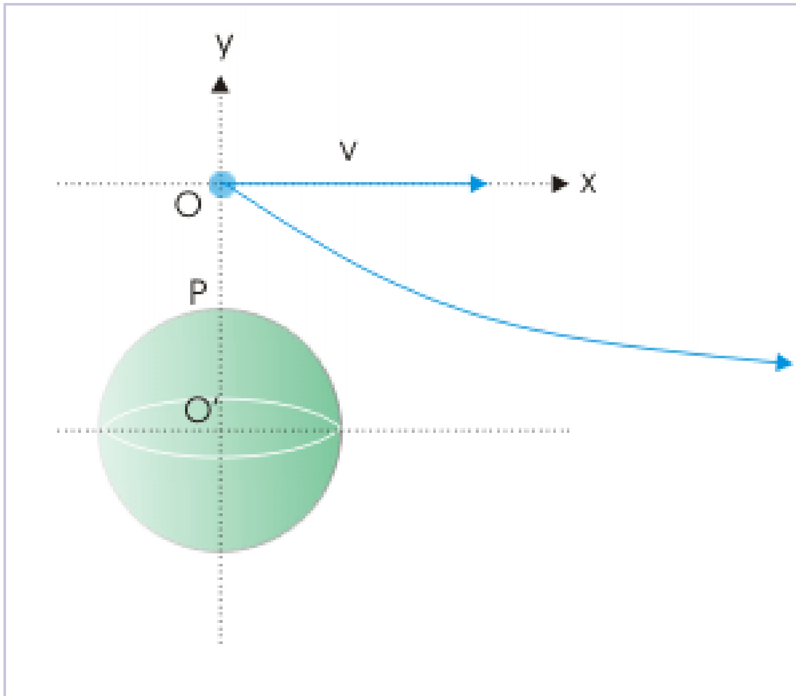

Figure 12.7: Projectile is projected with certain velocity in x-direction.

Chapter 13

Projection in gravitational field (application)[1]

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

13.1 Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the projection in gravitational field. The questions are categorized in terms of the characterizing features of the subject matter :

- Satellite
- Vertical projection
- Escape velocity

13.2 Satellite

13.2.1

Problem 1: A satellite of mass "m" is to be launched into an orbit around Earth of mass "M" and radius "R" at a distance "2R" from the surface. Find the minimum energy required to launch the satellite in the orbit.

Solution : The energy to launch the satellite should equal to difference of total mechanical energy of the system in the orbit and at Earth's surface. The mechanical energy of the satellite at the surface is only its potential energy. It is given by,

$$E_S = -\frac{GMm}{R}$$

On the other hand, satellite is placed at a total distance of R + 2R = 3R. The total mechanical energy of the satellite in the orbit is,

$$E_O = -\frac{GMm}{2r} = -\frac{GMm}{2X3R} = -\frac{GMm}{6R}$$

[1]This content is available online at <http://cnx.org/content/m15151/1.1/>.

Hence, energy required to launch the satellite is :

$$E = E_O - E_S$$

$$\Rightarrow E = -\frac{GMm}{6R} - \left(-\frac{GMm}{R}\right) = \frac{5GMm}{6R}$$

13.2.2

Problem 2: A satellite revolves around Earth of radius "R" with a speed "v". If rockets are fired to stop the satellite to make it standstill, then find the speed with which satellite will strike the Earth. Take g = 10 $m\ /\ s^2$.

Solution : When the seed of satellite is reduced to zero, it starts falling towards center of Earth. Let the velocity with which it strikes the surface be "v"' and distance between center of Earth and satellite be "r".

Applying conservation of energy :

$$\frac{1}{2}mv\prime^2 - \frac{GMm}{R} = 0 - \frac{GMm}{r}$$

For satellite, we know that kinetic energy is :

$$\frac{1}{2}mv^2 = \frac{GMm}{2r}$$

Also, we can express "GM" in terms of acceleration at the surface,

$$gR^2 = GM$$

Substituting these expressions in the equation of law of conservation of mechanical energy and rearranging,

$$\Rightarrow \frac{1}{2}mv\prime^2 = gRm - mv^2$$

$$\Rightarrow \frac{1}{2}v\prime^2 = gR - v^2$$

$$\Rightarrow v\prime = \sqrt{2\left(gR - v^2\right)}$$

$$\Rightarrow v\prime = \sqrt{\left(20R - 2v^2\right)}$$

13.3 Vertical projection

Problem 3: A particle is projected with initial speed equal to the orbital speed of a satellite near Earth's surface. If the radius of Earth is "R", then find the height to which the particle rises.

Solution : It is given that speed of projection is equal to orbital speed of a satellite near Earth's surface. The orbital speed of the satellite near Earth's surface is given by putting "r = R" in the expression of orbital velocity :

$$v = \sqrt{\left(\frac{GM}{R}\right)}$$

$$v^2 = \frac{GM}{R}$$

Since orbital velocity is less than escape velocity, the particle is returned to the surface after attaining a certain maximum height, "h". Applying conservation of energy, the height attained by projectile is obtained as :

$$h = \frac{v^2}{2g - \frac{v^2}{R}}$$

Substituting for "v" and "g", we have :

$$\Rightarrow h = \frac{\frac{GM}{R}}{\frac{2GM}{R^2} - \frac{GM}{R}}$$

$$\Rightarrow h = R$$

13.4 Escape velocity

13.4.1

Problem 4: A particle is fired with a velocity 16 km/s from the surface Earth. Find its velocity with which it moves in the interstellar space. Consider Earth's escape velocity as 11.2 km/s and neglect friction.

Solution : We observe here that initial velocity of the particle is greater than Earth's escape velocity. We can visualize this situation in terms of energy. The kinetic energy of the particle is used to (i) overcome the mechanical energy binding it to the gravitational influence of Earth and (ii) to move into interstellar space with a certain velocity.

Let "v", " v_e " and " v_i " be velocity of projection, escape velocity and velocity in the interstellar space respectively. Then, applying law of conservation of energy :

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}mv_i^2$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{GMm}{R} + \frac{1}{2}mv_i^2$$

Here, we have considered gravitational potential energy in the interstellar space as zero. Also, we know that kinetic energy corresponding to escape velocity is equal to the magnitude of gravitational potential energy of the particle on the surface. Hence,

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}mv_e^2 + \frac{1}{2}mv_i^2$$

$$\Rightarrow v_i^2 = v^2 - v_e^2$$

The escape velocity for Earth is 11.2 km/s. Putting values in the equation, we have :

$$\Rightarrow v_i^2 = (16)^2 - (11.2)^2$$

$$\Rightarrow v_i^2 = 256 - 125.44 = 130.56$$

$$\Rightarrow v_i = 11.43 \quad km/s$$

13.4.2

Problem 5: A satellite is orbiting near surface with a speed "v". What additional velocity is required to be imparted to the satellite so that it escapes Earth's gravitation. Consider, g = 10 m / s^2 and R = 6400 km.

Solution : The orbital speed of the satellite near Earth's surface is given by :

$$v = \sqrt{\left(\frac{GM}{R}\right)}$$

We can write this expression in terms of acceleration at the surface (g),

$$\Rightarrow v = \sqrt{\left(\frac{GM}{R}\right)} = \sqrt{\left(\frac{gR^2}{R}\right)} = \sqrt{(gR)}$$

On the other hand, escape velocity is given by :

$$v_e = \sqrt{(2gR)}$$

Hence, additional velocity to be imparted is difference of two speeds,

$$\Rightarrow v_e - v = \sqrt{(2gR)} - \sqrt{(gR)}$$

$$\Rightarrow v_e - v = \left(\sqrt{2} - 1\right) \sqrt{(gR)}$$

$$\Rightarrow v_e - v = \left(\sqrt{2} - 1\right) \sqrt{\left(10X6.4X10^6\right)}$$

$$v_e - v = 3.31X10^3 \quad m/s$$

Chapter 14

Two body system - linear motion[1]

"Two body" system represents the starting point for studying motion of celestial bodies, including Earth. In general, gravitational force is dominant for a pair of masses in such a manner that influence of all other bodies can be neglected as first approximation. In that case, we are left with an isolated "two body" system. The most important deduction of this simplifying assumption is that isolated system is free of external force. This means that "center of mass" of the isolated system in not accelerated.

In the solar system, one of the massive bodies is Sun and the other is one of the planets. In this case, Sun is relatively much larger than second body. Similarly, in Earth-moon system, Earth is relatively much larger than moon. On the other hand, bodies are of similar mass in a "binary stars" system. There are indeed various possibilities. However, we first need to understand the basics of the motion of isolated two bodies system, which is interacted by internal force of attraction due to gravitation. Specially, how do they hold themselves in space?

In this module, we shall apply laws of mechanics, which are based on Newton's laws of motion and Newton's law of gravitation. Most characterizing aspect of the motion is that two bodies, in question, move in a single plane, which contains their center of mass. What it means that the motion of two body system is coplanar.

[1]This content is available online at <http://cnx.org/content/m15152/1.3/>.

Two body system

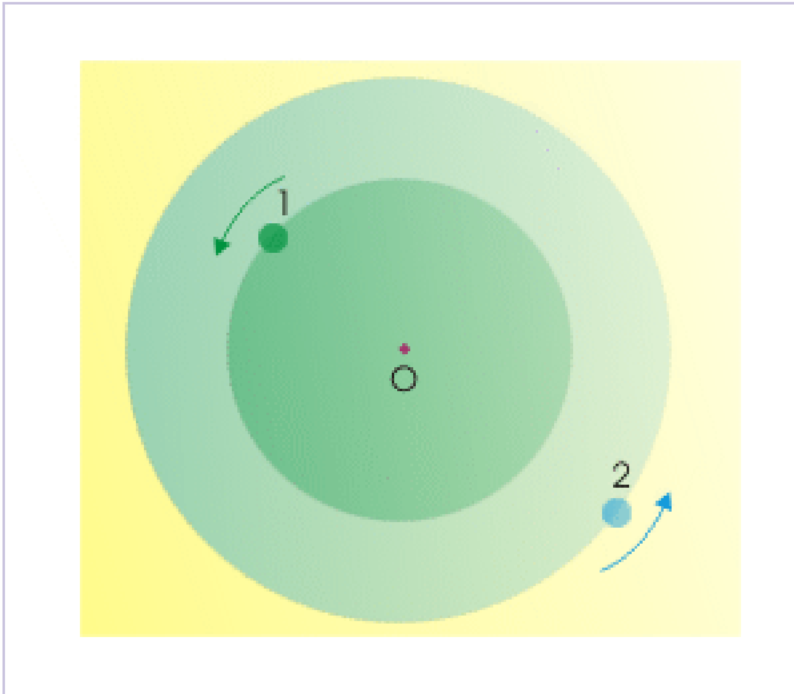

Figure 14.1: The plane of "one body and center of mass" and plane of "other body and center of mass" are in the same plane.

Newtonian mechanics provides a general solution in terms of trajectory of a conic section with different eccentricity. The trajectories like linear, circular, elliptical, parabolic, hyperbolic etc are subsets of this general solution with specific eccentricity. Here, we do not seek mathematical derivation of generalized solution of the motion. Rather, we want to introduce simpler trajectories like that of a straight line, circle etc. first and then interpret elliptical trajectory with simplifying assumptions. In this module, we shall limit ourselves to the motion of "two body" system along a straight line. We shall take up circular motion in the next module.

In a way, the discussion of motion of "two body" system is preparatory before studying Kepler's laws of planetary motion, which deals with specific case of elliptical trajectory.

NOTE: The general solution of two bodies system involves polar coordinates (as it suits the situation), vector algebra and calculus. In this module, however, we have retained rectangular coordinates for the most part with scalar derivation and limited our discussion to specific case of linear trajectory.

14.1 Straight line trajectory

This is simplest motion possible for "two body" system. The bodies under consideration are initially at rest. In this case, center of mass of two bodies is a specific point in the given reference. Also, it is to be noted that center of mass lies always between two bodies and not beyond them.

Since no external force is applied, the subsequent motion due to internal gravitational force does not change the position of center of mass in accordance with second law of motion. The bodies simply move towards each other such that center of mass remains at rest.

The two bodies move along a straight line joining their centers. The line of motion also also passes through center of mass. This has one important implication. The plane containing one body and "center of mass" and the plane containing other body and "center of mass" are same. It means that motions of two bodies are "coplanar" with line joining the centers of bodies and center of mass. The non-planar motions as shown in the figure below are not possible as motion is not along the line joining the centers of two bodies.

Two body system

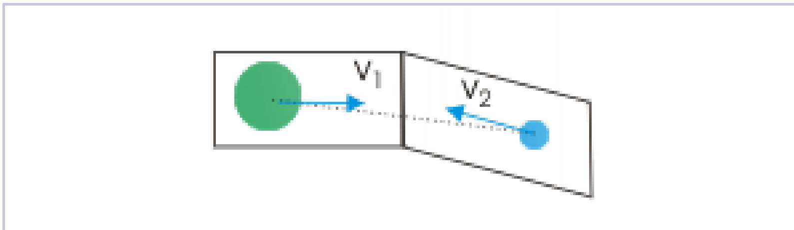

Figure 14.2: The motion of two body system can not be non-planar.

Let suscripts "1" and "2" denote two bodies. Also, let " r_1 ", " v_1 ", " a_1 " and " r_2 ", " v_2", " a_2 " be the magnitudes of linear distance from the center of mass, speeds and magnitudes of accelerations respectively of two bodies under consideration. Also let "center of mass" of the system is the origin of reference frame. Then, by definition of center of mass :

Two body system

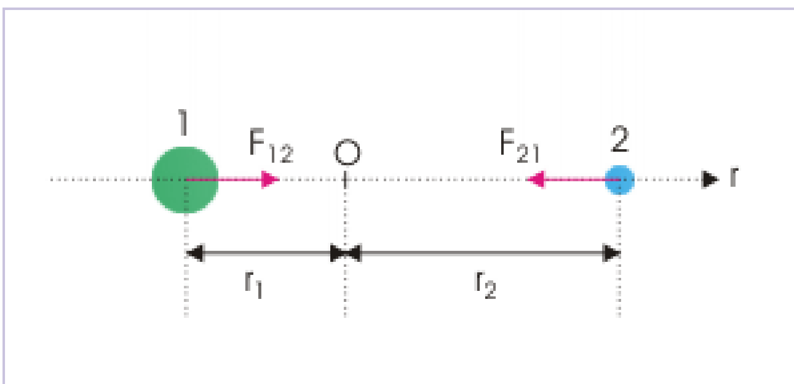

Figure 14.3: Center of mass is the origin of reference frame.

$$r_{cm} = \frac{-m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

But "center of mass" lies at the origin of the reference frame,

$$\Rightarrow r_{cm} = \frac{-m_1 r_1 + m_2 r_2}{m_1 + m_2} = 0$$

$$\Rightarrow m_1 r_1 = m_2 r_2$$

Taking first differentiation of position with respect to time, we have :

$$\Rightarrow v_{cm} = \frac{-m_1 v_1 + m_2 v_2}{m_1 + m_2} = 0$$

$$\Rightarrow m_1 v_1 = m_2 v_2$$

Taking first differentiation of velocity with respect to time, we have :

$$\Rightarrow a_{cm} = \frac{-m_1 a_1 + m_2 a_2}{m_1 + m_2} = 0$$

$$\Rightarrow m_1 a_1 = m_2 a_2$$

Considering only magnitude and combining with Newton's law of gravitation,

$$F_{12} = F_{21} = \frac{G m_1 m_2}{(r_1 + r_2)^2}$$

Since distance of bodies from center of mass changes with time, the gravitational force on two bodies is equal in magnitude at a given instant, but varies with time.

14.2 Newton's Second law of motion

We can treat "two body" system equivalent to "one body" system by stating law of motion in appropriate terms. For example, it would be interesting to know how force can be related to the relative acceleration with which two bodies are approaching towards each other. Again, we would avoid vector notation and only consider the magnitudes of accelerations involved. The relative acceleration is sum of the magnitudes of individual accelerations of the bodies approaching towards each other :

$$a_r = a_1 + a_2$$

According to Newton's third law, gravitational force on two bodies are pair of action and reaction and hence are equal in magnitude. The magnitude of force on each of the bodies is related to acceleration as :

$$F = m_1 a_1 = m_2 a_2$$

$$\Rightarrow m_1 a_1 = m_2 a_2$$

We can note here that this relation, as a matter of fact, is same as obtained using concept of center of mass. Now, we can write magnitude of relative acceleration of two bodies, " a_r ", in terms of individual accelerations is :

$$\Rightarrow a_r = a_1 + \frac{m_1 a_1}{m_2}$$

$$\Rightarrow a_r = a_1 \left(\frac{m_1 + m_2}{m_2} \right)$$

$$\Rightarrow a_1 = \frac{m_2 a_r}{m_1 + m_2}$$

Substituting in the Newton's law of motion,

$$\Rightarrow F = m_1 a_1 = \frac{m_1 m_2 a_r}{m_1 + m_2}$$

$$\Rightarrow F = \mu a_r$$

Where,

$$\Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2}$$

14.2.1 Reduced mass

The quantity given by the expression :

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is known as "reduced mass". It has the same unit as that of "mass". It represents the "effect" of two bodies, if we want to treat "two body" system as "one body" system.

In the nutshell, we can treat motion of "two body" system along a straight line as "one body" system, which has a mass equal to "μ" and acceleration equal to relative acceleration, " a_r ".

14.2.2 Velocity of approach

The two bodies are approaching towards each other. Hence, magnitude of velocity of approach is given by :

Velocity of approach

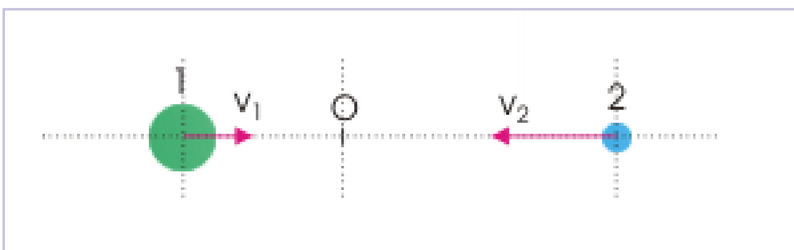

Figure 14.4: Two bodies are approaching each other along a straight line.

$$v_r = v_1 + v_2$$

This is the expression of the magnitude of velocity of approach. We can write corresponding vector equation for velocity of approach in relation to reference direction. In this module, however, we will avoid vector notation or vector interpretation to keep the discussion simplified.

14.3 Kinetic energy

The kinetic energy of the "two body" system is given as the sum of kinetic energy of individual bodies,

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

We can write this expression of kinetic energy in terms of relative velocity i.e. velocity of approach. For this, we need to express individual speeds in terms of relative speed as :

$$v_r = v_1 + v_2$$

But,

$$m_1 v_1 = m_2 v_2$$

$$\Rightarrow v_2 = \frac{m_1 v_1}{m_2}$$

Substituting in the expression of relative velocity,

$$\Rightarrow v_r = v_1 + \frac{m_1 v_1}{m_2}$$

$$\Rightarrow v_1 = \frac{m_2 v_r}{m_1 + m_2}$$

Similarly,

$$\Rightarrow v_2 = \frac{m_1 v_r}{m_1 + m_2}$$

Now, putting these expressions of individual speed in the equation of kinetic energy :

$$\Rightarrow K = \frac{m_1 m_2^2 v_r^2}{(m_1 + m_2)^2} + \frac{m_2 m_1^2 v_r^2}{(m_1 + m_2)^2}$$

$$\Rightarrow K = \frac{m_1 m_2 v_r^2}{(m_1 + m_2)^2} (m_1 + m_2)$$

$$\Rightarrow K = \frac{m_1 m_2 v_r^2}{m_1 + m_2}$$

$$\Rightarrow K = \frac{1}{2} \mu v_r^2$$

This result also indicates that we can treat "two body" system as "one body" system from the point of view of kinetic energy, as if the body has reduced mass of "μ" and speed equal to the magnitude of relative velocity, " v_r".

14.3.1 Example

Problem 1: Two masses " m_1 " and " m_2 " are initially at rest at a great distance. At a certain instant, they start moving towards each other, when released from their positions. Considering absence of any other gravitational field, calculate velocity of approach when they are at a distance "r" apart.

Solution : The bodies are at large distance in the beginning. There is no external gravitational field. Hence, we can consider initial gravitational energy of the system as zero (separated y infinite distance). Also, the bodies are at rest in the beginning. The initial kinetic energy is also zero. In turn, initial mechanical energy of the system is zero.

Let " v_r " be the velocity of approach, when the bodies are at a distance "r" apart. Applying conservation of mechanical energy, we have :

$$K_i + U_i = K_f + U_f$$

$$\Rightarrow 0 + 0 = \frac{1}{2}\mu v_r^2 - \frac{Gm_1m_2}{r}$$

$$\Rightarrow v_r^2 = \frac{2Gm_1m_2}{\mu r} = \frac{2Gm_1m_2\left(m_1 + m_2\right)}{m_1m_2r}$$

$$\Rightarrow v_r = \sqrt{\left\{\frac{2G\left(m_1 + m_2\right)}{r}\right\}}$$

14.4 Conclusions

From the discussion above, we conclude the followings about the motion of "two body" system along a straight line :

1: Each body follows a straight line trajectory.

2: The line joining centers of two bodies pass through center of mass.

3: The planes of two motions are in the same plane. In other words, two motions are coplanar.

4: Magnitude of gravitational force is same for two bodies, but they vary as the distance between them changes.

5: We can treat "two body" system equivalent to "one body" system by using concepts of (i) reduced mass "μ" (ii) relative velocity, " v_r " and (iii) relative acceleration " a_r ".

Chapter 15

Two body system - circular motion[1]

The trajectory of two body system depends on the initial velocities of the bodies and their relative mass. If the mass of the bodies under consideration are comparable, then bodies move around their "center of mass" along two separate circular trajectories. This common point about which two bodies revolve is also known as "barycenter".

In order to meet the requirement imposed by laws of motion and conservation laws, the motion of two bodies executing circular motion is constrained in certain ways.

15.1 Circular trajectory

Since external force is zero, the acceleration of center of mass is zero. This is the first constraint. For easy visualization of this constraint, we consider that center of mass of the system is at rest in a particular reference frame.

Now, since bodies are moving along two circular paths about "center of mass", their motions should be synchronized in a manner so that the length of line, joining their centers, is a constant . This is required; otherwise center of mass will not remain stationary in the chosen reference. Therefore, the linear distance between bodies is a constant and is given by :

[1]This content is available online at <http://cnx.org/content/m15153/1.4/>.

Two body system - circular motion

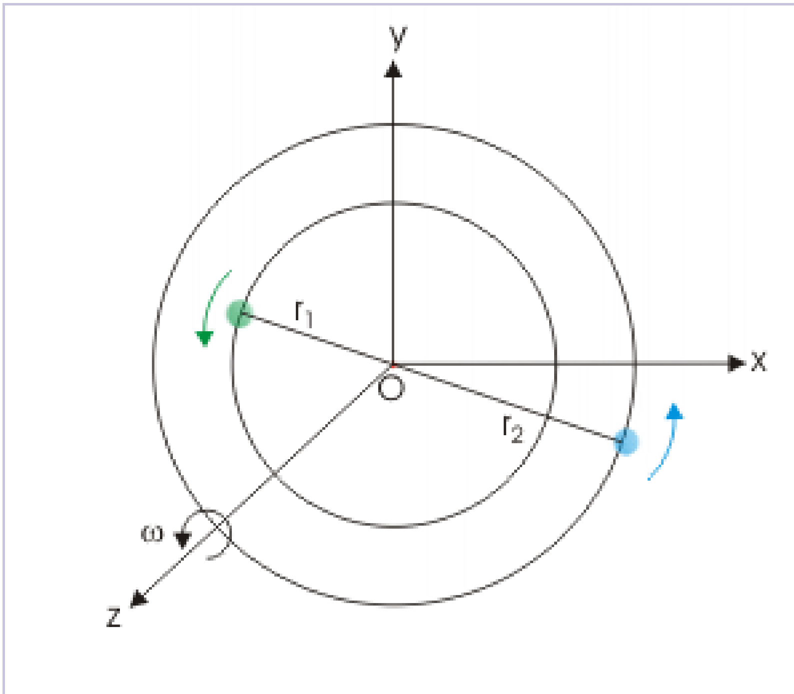

Figure 15.1: Each body moves around center of mass.

$$r = r_1 + r_2$$

Now this condition can be met even if two bodies move in different planes. However, there is no external torque on the system. It means that the angular momentum of the system is conserved. This has an important deduction : the plane of two circular trajectories should be same.

Mathematically, we can conclude this, using the concept of angular momentum. We know that torque is equal to time rate of change of angular momentum,

$$\frac{dL}{dt} = r \times F$$

But, external torque is zero. Hence,

$$\Rightarrow r \times F = 0$$

It means that "**r**" and "**F**" are always parallel. It is only possible if two planes of circles are same. We, therefore, conclude that motions of two bodies are coplanar. For coplanar circular motion, center of mass is given by definition as :

Two body system - circular motion

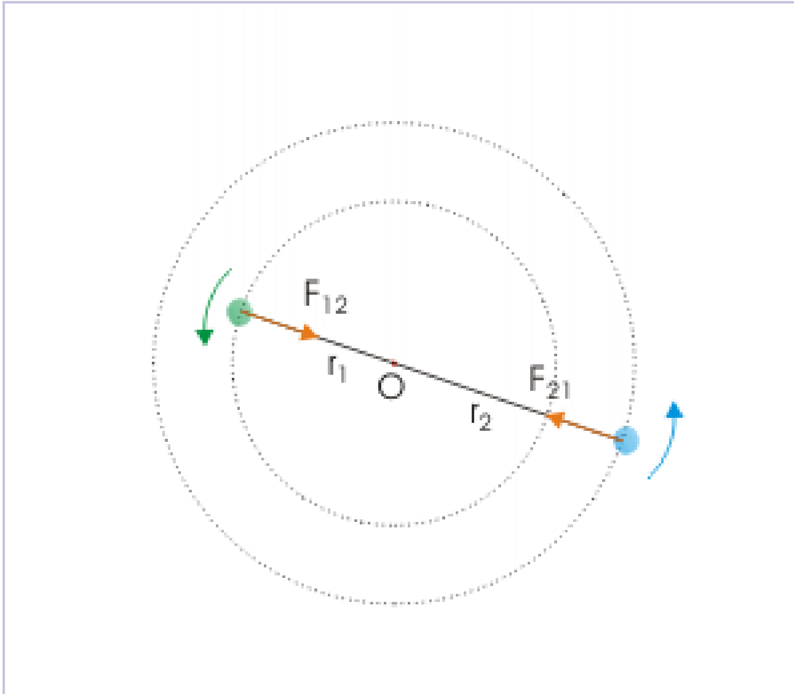

Figure 15.2: Each body moves around center of mass.

$$r_{cm} = \frac{-m_1 r_1 + m_2 r_2}{m_1 + m_2} = 0$$

$$\Rightarrow m_1 r_1 = m_2 r_2$$

Taking first differentiation with respect to time, we have :

$$\Rightarrow m_1 v_1 = m_2 v_2$$

Now dividing second equation by first,

$$\Rightarrow \frac{m_1 v_1}{m_1 r_1} = \frac{m_2 v_2}{m_2 r_2}$$

$$\Rightarrow \frac{v_1}{r_1} = \frac{v_2}{r_2}$$

$$\Rightarrow \omega_1 = \omega_2 = \omega \quad (say)$$

It means that two bodies move in such a manner that their angular velocities are equal.

Two body system - circular motion

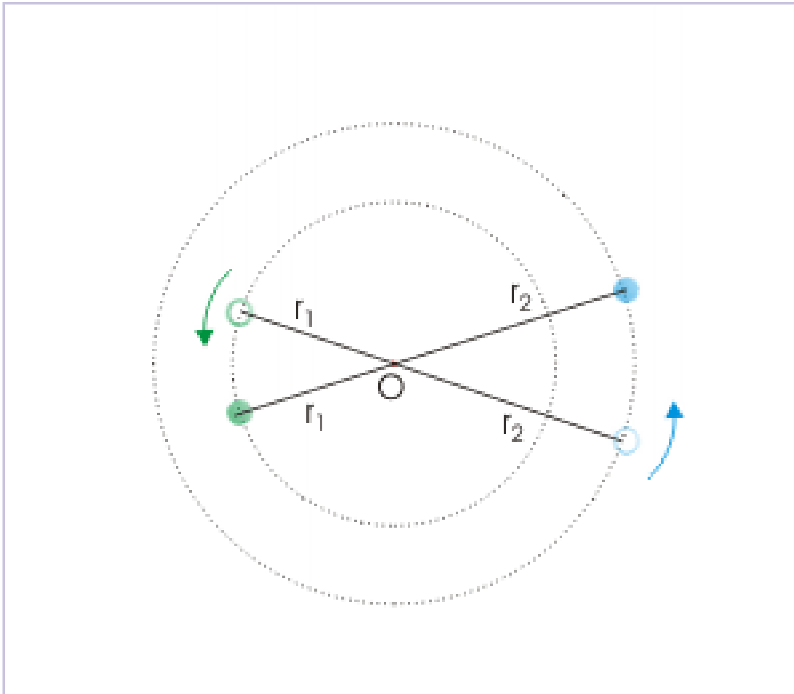

Figure 15.3: Both bodies move with same angular velocity.

15.1.1 Gravitational force

The gravitational force on each of the bodies is constant and is given by :

$$F = \frac{Gm_1 m_2}{(r_1 + r_2)^2} = \frac{Gm_1 m_2}{r^2}$$

Since gravitational force provides for the requirement of centripetal force in each case, it is also same in two cases. Centripetal force is given by :

$$F_C = m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{Gm_1 m_2}{r^2}$$

15.2 Angular velocity

Each body moves along a circular path. The gravitational force on either of them provides the centripetal force required for circular motion. Hence, centripetal force is :

$$m_1 r_1 \omega^2 = \frac{Gm_1 m_2}{(r_1 + r_2)^2}$$

$$\Rightarrow \omega^2 = \frac{Gm_2}{r_1 (r_1 + r_2)^2}$$

Let the combined mass be "M". Then,

$$M = m_1 + m_2$$

Using relation $m_1 r_1 = m_2 r_2$, we have :

$$\Rightarrow M = \frac{m_2 r_2}{r_1} + m_2 = m_2 \left(\frac{r_1 + r_2}{r_1} \right)$$

$$\Rightarrow m_2 = \frac{M r_1}{r_1 + r_2}$$

Substituting in the equation, involving angular velocity,

$$\Rightarrow \omega^2 = \frac{G M r_1}{r_1 (r_1 + r_2)^3} = \frac{GM}{r^3}$$

$$\Rightarrow \omega = \sqrt{\left(\frac{GM}{r^3} \right)}$$

This expression has identical form as for the case when a body revolves around another body at rest along a circular path (compare with "Earth – satellite" system). Here, combined mass "M" substitutes for the mass of heavier mass at the center and sum of the linear distance replaces the radius of rotation.

The linear velocity is equal to the product of the radius of circle and angular velocity. Hence,

$$v_1 = \omega r_1$$

$$v_2 = \omega r_2$$

15.2.1 Time period

We can easily find the expression for time period of revolution as :

$$T = \frac{2\pi}{\omega} = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{(GM)}}$$

This expression also has the same form as for the case when a body revolves around another body at rest along a circular path (compare with "Earth – satellite" system). Further squaring on either side, we have :

$$\Rightarrow T^2 \propto r^3$$

15.3 Moment of inertia

Here, we set out to find moment of inertia of the system about the common axis passing through center of mass and perpendicular to the plane of rotation. For this, we consider each of the bodies as point mass. Note that two bodies are rotating about a common axis with same angular velocity. Clearly, MI of the system is :

$$I = m_1 r_1^2 + m_2 r_2^2$$

We can express individual distance in terms of their sum using following two equations,

$$r = r_1 + r_2$$

$$m_1 r_1 = m_2 r_2$$

Substituting for " r_1 " in the equation or "r", we have :

$$\Rightarrow r = \frac{m_2 r_2}{m_1} + r_2 = r_2 \left(\frac{m_1 + m_2}{m_1} \right)$$

$$\Rightarrow r_2 = \frac{r m_1}{m_1 + m_2}$$

Similarly, we can express, " r_1 " as :

$$\Rightarrow r_1 = \frac{r m_2}{m_1 + m_2}$$

Substituting for " r_1 " and " r_2 " in the expression of moment of inertia,

$$\Rightarrow I = \frac{m_1 m_2^2 r^2}{(m_1 + m_2)^2} + \frac{m_2 m_1^2 r^2}{(m_1 + m_2)^2}$$

$$\Rightarrow I = \frac{m_1 m_2 r^2}{(m_1 + m_2)^2} X (m_1 + m_2)$$

$$\Rightarrow I = \frac{m_1 m_2 r^2}{m_1 + m_2}$$

$$\Rightarrow I = \mu r^2$$

This expression is similar to the expression of momemnt of inertia of a particle about an axis at a perpendicualr distance, "r". It is, therefore, clear that "Two body" system orbiting around center of mass can be treated as "one body" system by using concepts of net distance "r" and reduced mass "μ".

15.4 Angular momentum

The bodies move about the same axis with the same sense of rotation. The angular momentum of the system, therefore, is algebraic sum of individual angular momentums.

$$L = L_1 + L_2 = m_1 r_1^2 \omega + m_2 r_2^2 \omega$$

Substituting for " r_1 " and " r_2 " with expressions as obtained earlier,

$$\Rightarrow L = \frac{m_1 m_2^2 r^2 \omega}{(m_1 + m_2)^2} + \frac{m_2 m_1^2 r^2 \omega}{(m_1 + m_2)^2}$$

$$\Rightarrow L = \frac{m_1 m_2 r^2 \omega}{(m_1 + m_2)^2} X (m_1 + m_2)$$

$$\Rightarrow L = \frac{m_1 m_2 r^2 \omega}{(m_1 + m_2)}$$

$$\Rightarrow L = \mu r^2 \omega$$

This expression is similar to the expression of angular momemntum of a particle about an axis at a perpendicualr distance, "r". Once again, we see that "Two body" system orbiting around center of mass can be treated as "one body" system by using concepts of net distance "r" and reduced mass "μ".

15.5 Kinetic energy

The kinetic energy of the system is equal to the algebraic sum of the kinetic energy of the individual body. We write expression of kinetic energy in terms of angular velocity – not in terms of linear velocity. It is so because angular velocity is same for two bodies and can, therefore, be used to simplify the expression for kinetic energy. Now, kinetic energy of the system is :

$$K = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2$$

Substituting for " r_1 " and " r_2 " with expressions as obtained earlier,

$$\Rightarrow K = \frac{m_1 m_2^2 r^2 \omega^2}{2{(m_1 + m_2)}^2} + \frac{m_2 m_1^2 r^2 \omega^2}{2{(m_1 + m_2)}^2}$$

$$\Rightarrow K = \frac{m_1 m_2 r^2 \omega^2}{2{(m_1 + m_2)}^2} X \left(m_1 + m_2\right)$$

$$\Rightarrow K = \frac{m_1 m_2 r^2 \omega^2}{2\left(m_1 + m_2\right)}$$

$$\Rightarrow K = \frac{1}{2} \mu r^2 \omega^2$$

This expression of kinetic energy is also similar to the expression of kinetic energy of a particle rotating about an axis at a perpendicualr distance, "r". Thus, this result also substantiates equivalence of "Two body" system as "one body" system, using concepts of net distance "r" and reduced mass "μ".

15.6 Example

Problem 1 : In a binary star system, two stars of "m" and "M" move along two circular trajectories. If the distance between stars is "r", then find the total mechanical energy of the system. Consider no other gravitational influence on the system.

Solution : Mechanical energy of the system comprises of potential and kinetic energy. Hence,

$$E = \frac{1}{2} \mu r^2 w^2 - \frac{GMm}{r}$$

We know that angular velocity for "two body" system in circular motion is given by :

$$\Rightarrow \omega = \sqrt{\left\{\frac{G\left(M + m\right)}{r^3}\right\}}$$

Also, reduced mass is given by :

$$\mu = \frac{Mm}{M + m}$$

Putting in the expression of mechanical energy,

$$\Rightarrow E = \frac{mMr^2 G\left(m + M\right)}{2\left(m + M\right) r^3} - \frac{GMm}{r}$$

$$\Rightarrow E = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$\Rightarrow E = -\frac{GMm}{2r}$$

15.7 Conclusions

Thus, we conclude the following :

1: Each body follows a circular path about center of mass.

2: The line joining centers of two bodies pass through center of mass.

3: The planes of two motions are in the same plane. In other words, two motions are coplanar.

4: The angular velocities of the two bodies are equal.

5: The linear distance between two bodies remains constant.

6: Magnitude of gravitational force is constant and same for two bodies.

7: Magnitude of centripetal force required for circular motion is constant and same for two bodies.

8: Since linear velocity is product of angular velocity and distance from the center of revolution, it may be different if the radii of revolutions are different.

9: We can treat two body system with an equivalent one body system by using concepts of (i) combined mass, "M", (ii) net distance "r" and (iii) reduced mass "μ".

Two body system - circular motion

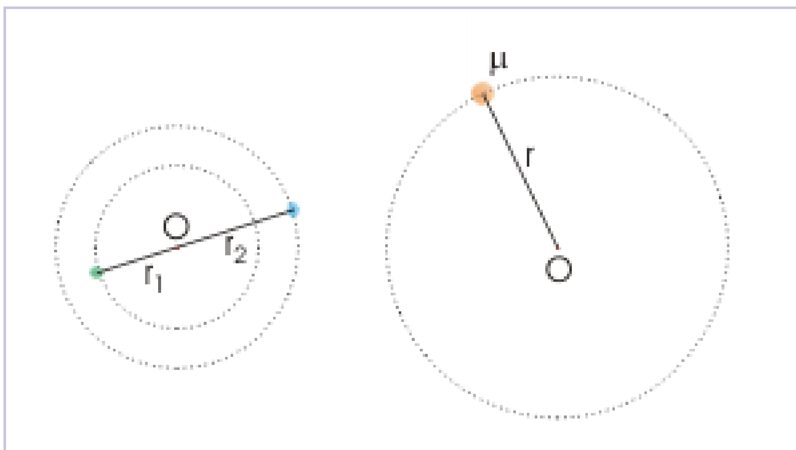

Figure 15.4: Two body system as equivalent to one body system.

Chapter 16

Planetary motion[1]

The trajectory of motion resulting from general solution of "two body" system is a conic section. Subject to initial velocities and relative mass, eccentricity of conic section can have different values. We interpret a conic section for different eccentricity to represent different types of trajectories. We have already discussed straight line and circular trajectories. In this module, we shall discuss elliptical trajectory, which is the trajectory of a planet in the solar system.

In a general scenario of "two body system", involving elliptical trajectory, each body revolves around the common "center of mass" called "barycenter". The two elliptical paths intersect, but bodies are not at the point of intersection at the same time and as such there is no collision.

[1] This content is available online at <http://cnx.org/content/m15186/1.10/>.

Two body system

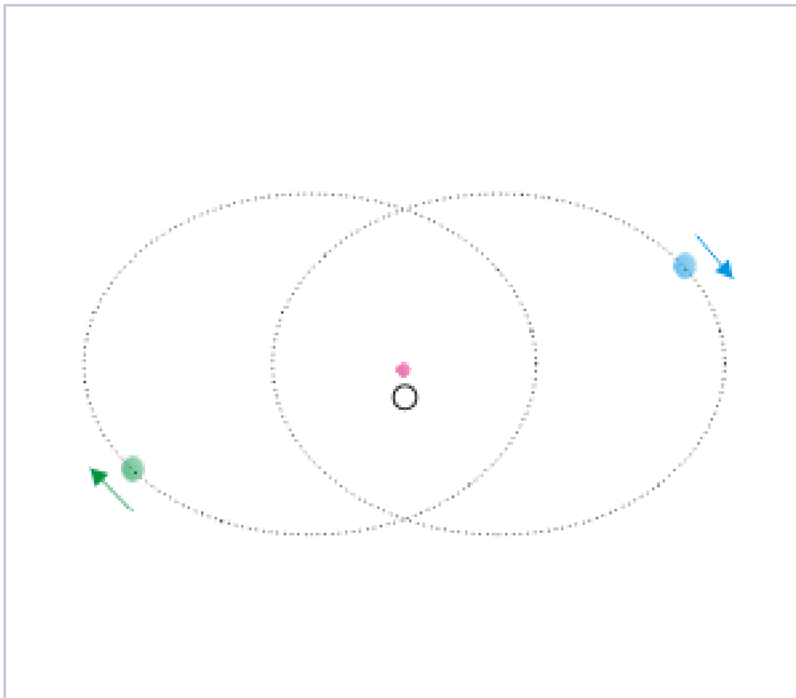

Figure 16.1: Elliptical trajectories

In this module, however, we shall keep our focus on the planetary motion and Kepler's planetary laws. We are basically seeking to describe planetary motion – particularly that in our solar system. The trajectory of planet is elliptical with one qualification. The Sun, being many times heavier than the planets, is almost at rest in the reference frame of motion. It lies at one of the foci of the elliptical path of the planets around it. Here, we assume that center of mass is about same as the center of Sun. Clearly, planetary motion is a special case of elliptical motion of "two body system" interacted by mutual attraction.

We should, however, be aware that general solution of planetary motion involves second order differential equation, which is solved using polar coordinates.

16.1 Ellipse

We need to learn about the basics of elliptical trajectory and terminology associated with it. It is important from the point of view of applying laws of Newtonian mechanics. We shall, however, be limited to the basics only.

16.1.1 Conic section

Conic section is obtained by the intersection of a plane with a cone. Two such intersections, one for a circle and one for an ellipse are shown in the figure.

Conic sections

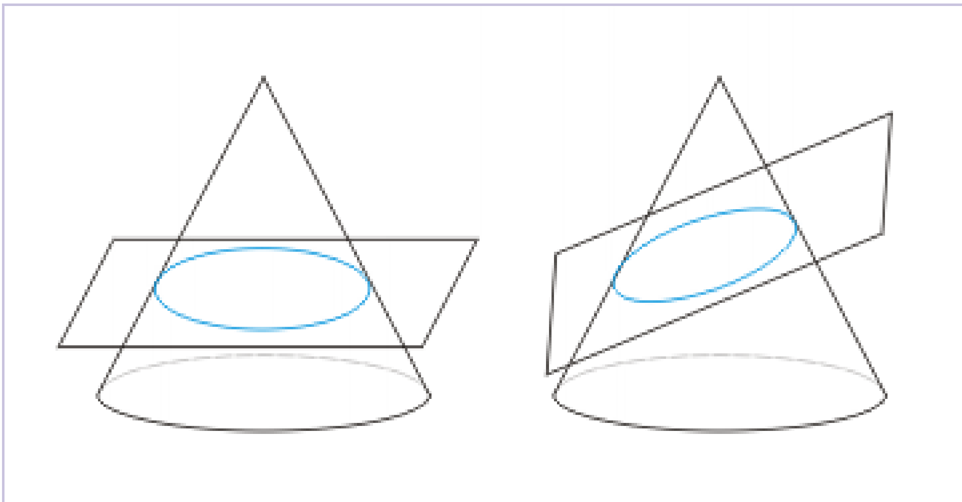

Figure 16.2: Two conic sections representing a circle and an ellipse are shown.

16.1.2 Elliptical trajectory

Here, we recount the elementary geometry of an ellipse in order to understand planetary motion. The equation of an ellipse centered at the origin of a rectangular coordinate (0,0) is :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where "a" is semi-major axis and "b" is semi-minor axis as shown in the figure.

Ellipse

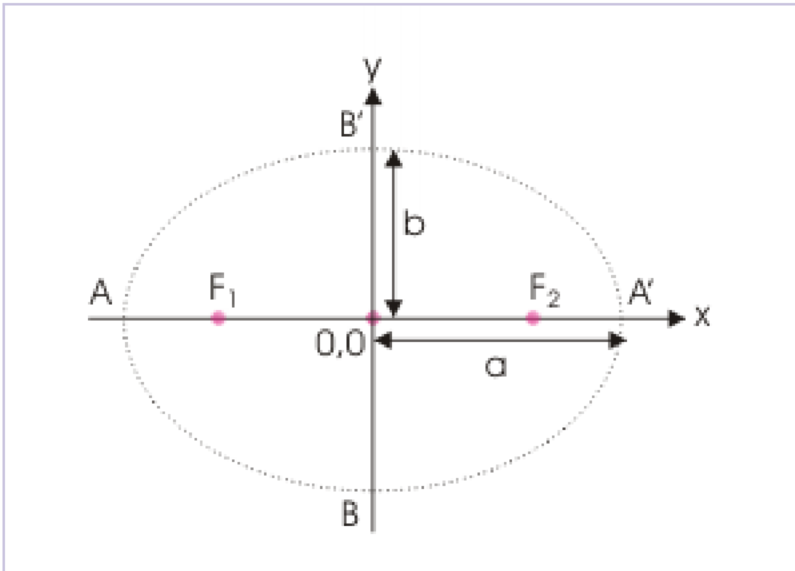

Figure 16.3: Semi major and minor axes of an ellipse

Note that " F_1 " and " F_2 " are two foci of the ellipse.

16.1.3 Eccentricity

The eccentricity of a conic section is measure of "how different it is from a circle". Higher the eccentricity, greater is deviation. The eccentricity (e) of a conic section is defined in terms of "a" and "b" as :

$$e = \sqrt{\left(1 - \frac{kb^2}{a^2}\right)}$$

where "k" is 1 for an ellipse, 0 for parabola and -1 for hyperbola. The values of eccentricity for different trajectories are as give here :

1. The eccentricity of a straight line is 1, if we consider b=0 for the straight line.
2. The eccentricity of an ellipse falls between 0 and 1.
3. The eccentricity of a circle is 0
4. The eccentricity of a parabola is 1.
5. The eccentricity of a hyperbola is greater than 1.

16.1.4 Focal points

Focal points (F_1 and F_2) lie on semi major axis at a distance from the origin given by

Focal points

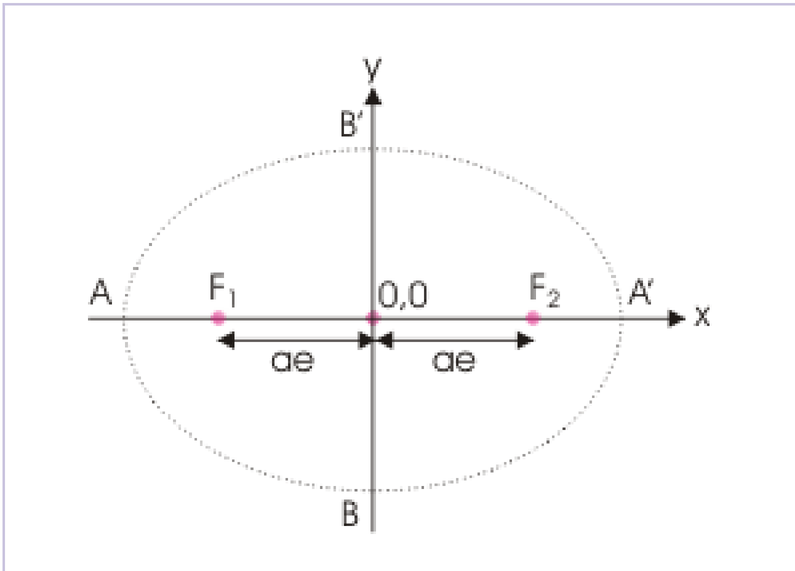

Figure 16.4: Focal distances from the center of ellipse

$$f = ae$$

The focus of an ellipse is at a distance "ae" from the center on the semi-major axis. Area of the ellipse is "πab".

16.1.5 Semi latus rectum

Semi latus rectum is equal to distance between one of the foci and ellipse as measured along a line perpendicular to the major axis. This is shown in the figure.

Semi latus rectum

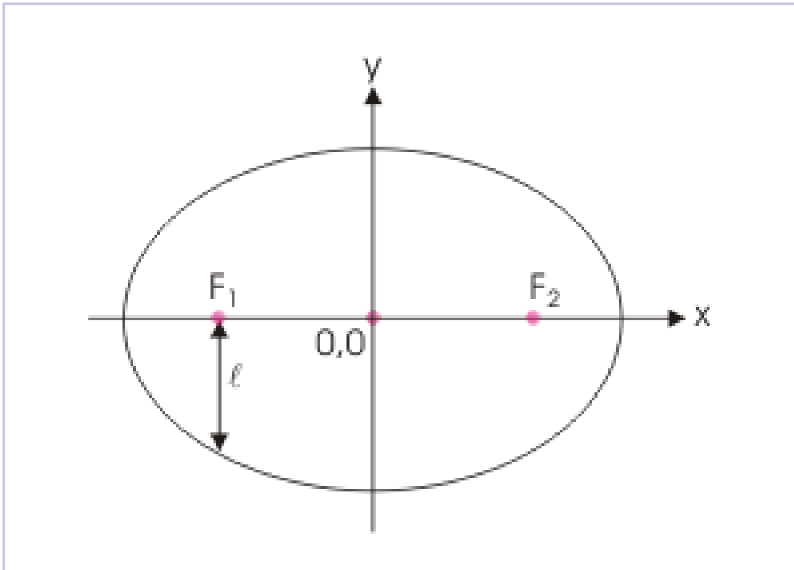

Figure 16.5: Semi latus rectum is perpendicular distance as shown in the figure.

For an ellipse, Semi latus rectum has the expression in terms of "a" and "b" as :

$$\ell = \frac{b^2}{a}$$

We can also express the same involving eccentricity as :

$$\Rightarrow \ell = a\left(1 - e^2\right)$$

16.2 Solar system

The solar system consists of Sun and its planets. The reason they are together is gravitation. The mass of the planet is relatively small with respect to Sun. For example, Earth compares about 10^5 times smaller in mass with respect to Sun :

Mass of Earth :

$$5.98X10^{24} \quad kg$$

Mass of Sun :

$$1.99X10^{30} \quad kg$$

The planetary motion, therefore, fits nicely with elliptical solution obtained from consideration of mechanics. Sun, being many times heavier, appears to be at the "center of mass" of the system i.e. at one of the foci, while planets revolve around it in elliptical orbits of different eccentricities.

16.2.1 Equation in polar coordinates

Polar coordinates generally suite geometry of ellipse. The figure shows the polar coordinates of a point on the ellipse. It is important to note that one of foci serves as the origin of polar coordinates, whereas the other focus lies on the negative x-axis. For this reason orientation of x-axis is reversed in the figure. The angle is measured anti-clockwise and the equation of ellipse in polar coordinates is :

Equation in polar coordinate

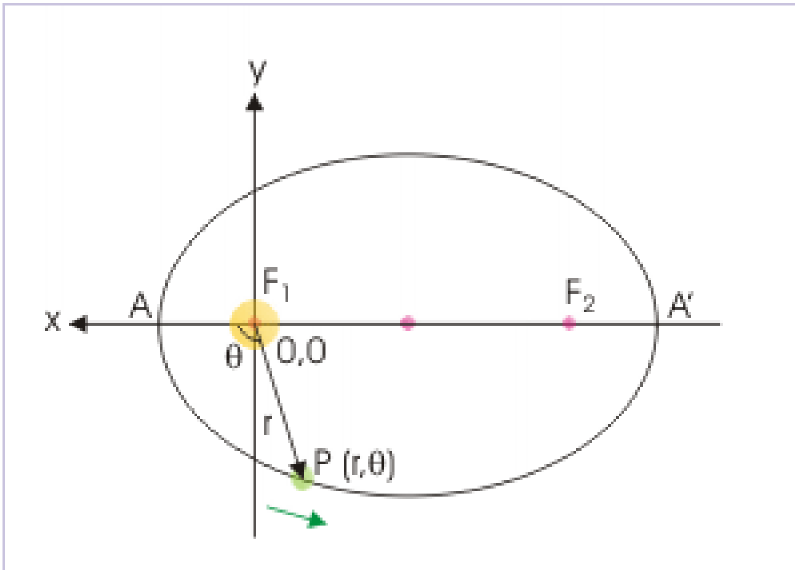

Figure 16.6: Second focus lies on negative x-axis.

$$r = \frac{\ell}{1 + e\cos\theta}$$

Substituting expression for semi latus rectum

$$\Rightarrow r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

16.2.1.1 Perihelion distance

Perihelion position corresponds to minimum distance between Sun and planet. If we consider Sun to be at one focus (say F_1), then perihelion distance is " $F_1 A$ " as shown in the figure. We can see that angle $\theta = 0^\circ$ for this position.

$$\Rightarrow r_{\min} = \frac{a\left(1 - e^2\right)}{1 + e} = a\left(1 - e\right)$$

Minimum and maximum distance

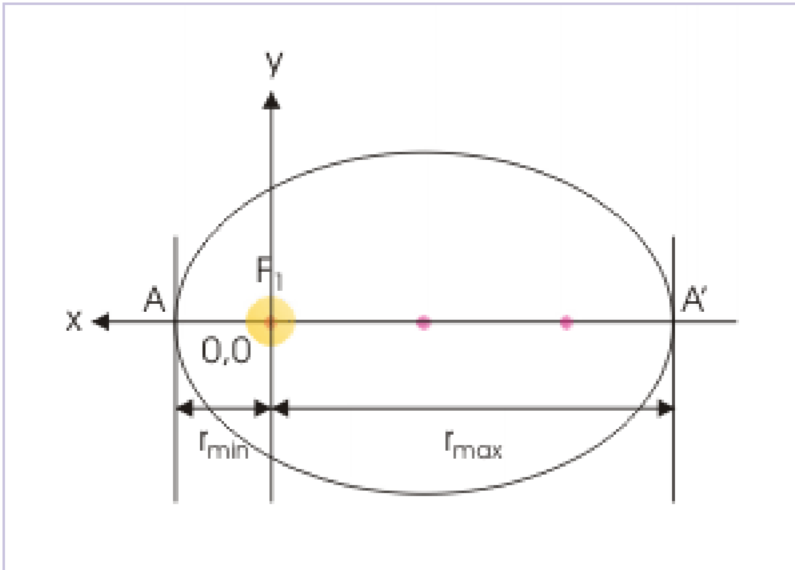

Figure 16.7: Positions correspond to perihelion and aphelion positions.

From the figure also, it is clear that minimum distance is equal to “a - ae= a(1-e)”.

16.2.1.2 Aphelion distance

Aphelion position corresponds to maximum distance between Sun and planet. If we consider Sun to be at one focus (say F_1), then perihelion distance is " F_1A' " as shown in the figure. We can see that angle $\theta = 180\,^\circ$ for this position.

$$\Rightarrow r_{\text{max}} = \frac{a\left(1-e^2\right)}{1-e} = a\left(1+e\right)$$

From the figure also, it is clear that maximum distance is equal to “a + ae= a(1+e)”.

We can also prove that the semi-major axis, “a” is arithmetic mean, whereas semi-minor axis, “b”, is geometric mean of “ r_{min} ” and “ r_{max}”.

16.3 Description of planetary motion

We can understand planetary motion by recognizing important aspects of motion like force, velocity, angular momentum, energy etc. The first important difference to motion along circular path is that linear distance between Sun and planet is not constant. The immediate implication is that gravitation force is not constant. It is maximum at perihelion position and minimum at aphelion position.

If “R” is the radius of curvature at a given position on the elliptical trajectory, then centripetal force equals gravitational force as given here :

$$\frac{mv^2}{R} = m\omega^2 R = \frac{GMm}{r^2}$$

Where “M” and “m” are the mass of Sun and Earth; and “r” is the linear distance between Sun and Earth.

Except for parameters "r" and "R", others are constant in the equation. We note that radii of curvature at perihelion and aphelion are equal. On the other hand, centripetal force is greatest at perihelion and least at aphelion. From the equation above, we can also infer that both linear and angular velocities of planet are not constant.

16.3.1 Angular momentum

The angular velocity of the planet about Sun is not constant. However, as there is no external torque working on the system, the angular momentum of the system is conserved. Hence, angular momentum of the system is constant unlike angular velocity.

The description of motion in angular coordinates facilitates measurement of angular momentum. In the figure below, linear momentum is shown tangential to the path in the direction of velocity. We resolve the linear momentum along the parallel and perpendicular to radial direction. By definition, the angular momentum is given by :

Angular momentum

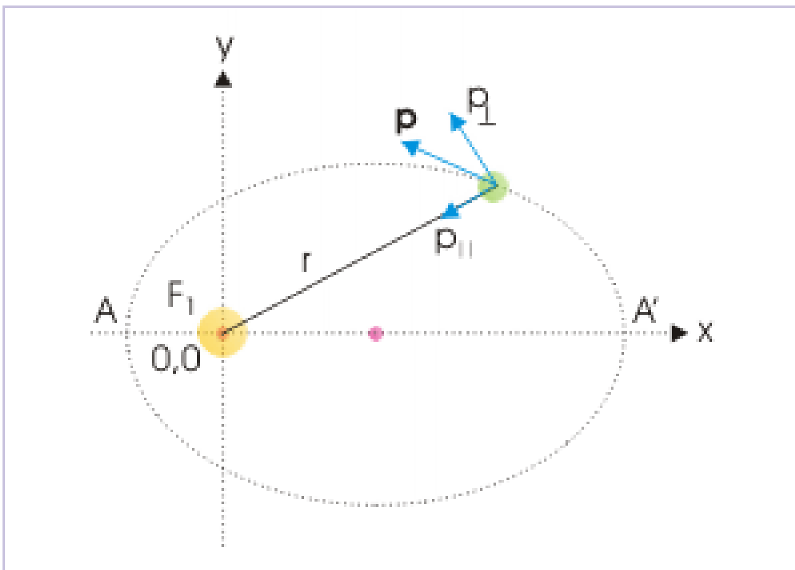

Figure 16.8: Angular momentum of the system is constant.

$$L = rXp_\perp$$

$$L = rXp_\perp = rmv_\perp = rmX\omega r = m\omega r^2$$

Since mass of the planet "m" is constant, it emerges that the term " ωr^2 " is constant. It clearly shows that angular velocity (read also linear velocity) increases as linear distance between Sun and Earth decreases and vice versa.

16.3.2 Maximum and minimum velocities

Maximum velocity corresponds to perihelion position and minimum to aphelion position in accordance with maximum and minimum centripetal force at these positions. We can find expressions of minimum and maximum velocities, using conservation laws.

Maximum and minimum velocities

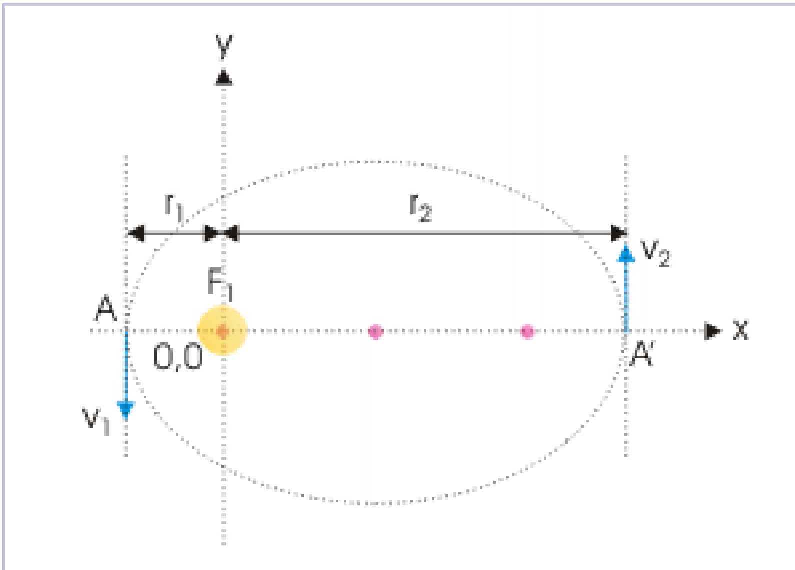

Figure 16.9: Velocities at these positions are perpendicular to semi major axis.

Let " r_1 " and " r_2 " be the minimum and maximum distances, then :

$$r_1 = a\,(1-e)$$

$$r_2 = a\,(1+e)$$

We see that velocities at these positions are perpendicular to semi major axis. Applying conservation of angular momentum,

$$L = r_1 m v_1 = r_2 m v_2$$

$$r_1 v_1 = r_2 v_2$$

Applying conservation of energy, we have :

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2}$$

Substituting for " v_2 ", " r_1 " and " r_2 ", we have :

$$v_1 = v_{\max} = \sqrt{\{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

$$v_2 = v_{\min} = \sqrt{\{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

16.3.3 Energy of Sun-planet system

As no external force is working on the system and there is no non-conservative force, the mechanical energy of the system is conserved. We have derived expression of linear velocities at perihelion and aphelion positions in the previous section. We can, therefore, find out energy of "Sun-planet" system by determining the same at either of these positions.

Let us consider mechanical energy at perihelion position. Here,

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{r_1}$$

Substituting for velocity and minimum distance, we have :

$$\Rightarrow E = \frac{mGM\,(1+e)}{2a\,(1+e)} - \frac{GMm}{a\,(1-e)}$$

$$\Rightarrow E = \frac{mGM}{a\,(1-e)}\left(\frac{1+e}{2} - 1\right)$$

$$\Rightarrow E = \frac{mGM}{a\,(1-e)}\left(\frac{e-1}{2}\right)$$

$$\Rightarrow E = -\frac{GMm}{2a}$$

We see that expression of energy is similar to that of circular trajectory about a center with the exception that semi major axis "a" replaces the radius of circle.

16.3.4 Kepler's laws

Johannes Kepler analyzed Tycho Brahe's data and proposed three basic laws that govern planetary motion of solar system. The importance of his laws lies in the fact that he gave these laws long before Newton's laws of motion and gravitation. The brilliance of the Kepler's laws is remarkable as his laws are consistent with Newton's laws and conservation laws.

Kepler proposed three laws for planetary motion. First law tells about the nature of orbit. Second law tells about the speed of the planet. Third law tells about time period of revolution.

- Law of orbits
- Law of velocities
- Law of time periods

16.3.4.1 Law of orbits

The first law (law of orbits) is stated as :

Definition 16.1: Law of orbits
The orbit of every planet is an ellipse with the sun at one of the foci.

This law describes the trajectory of a planet, which is an ellipse – not a circle. We have seen that application of mechanics also provides for elliptical trajectory. Only additional thing is that solution of mechanics yields possibilities of other trajectories as well. Thus, we can conclude that Kepler's law of orbit is consistent with Newtonian mechanics.

We should, however, note that eccentricity of elliptical path is very small for Earth (0.0167) and large for Mercury (0.206) and Pluto (0.25).

16.3.4.2 Law of velocities

Law of velocities is a statement of comparative velocities of planets at different positions along the elliptical path.

Definition 16.2: Law of velocities

The line joining a planet and Sun sweeps equal area in equal times in the planet's orbit.

This law states that the speed of the planet is not constant as generally might have been conjectured from uniform circular motion. Rather it varies along its path. A given area drawn to the focus is wider when it is closer to Sun. From the figure, it is clear that planet covers smaller arc length when it is away and a larger arc length when it is closer for a given orbital area drawn from the position of Sun. It means that speed of the planet is greater at positions closer to the Sun and smaller at positions away from the Sun.

Equal area swept in equal time

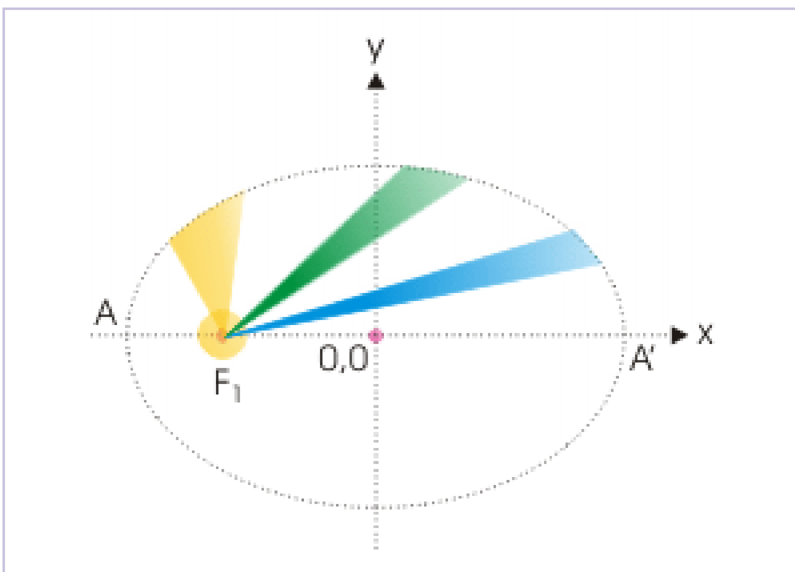

Figure 16.10: Speed of the planet is greater at positions closer to the Sun and smaller at positions away from the Sun.

Further on close examination, we find that Kepler's second law, as a matter of fact, is an statement of the conservations of angular momentum. In order to prove this, let us consider a small orbital area as shown in the figure.

Area swept

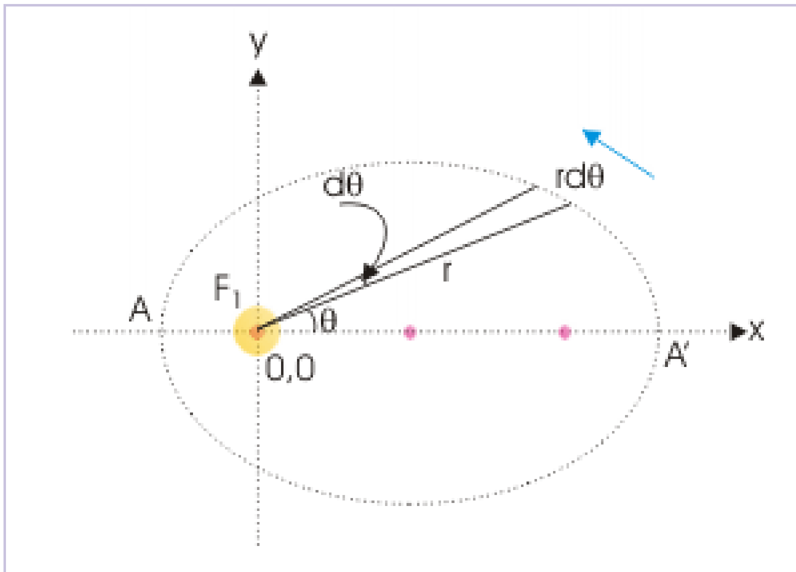

Figure 16.11: Time rate of area is statement of conservation of angular momentum.

$$\Delta A = \frac{1}{2} X \text{Base} X \text{Height} = \frac{1}{2} r \Delta\theta r = \frac{1}{2} r^2 \Delta\theta$$

For infinitesimally small area, the "area speed" of the planet (the time rate at which it sweeps orbital area drawn from the Sun) is :

$$\Rightarrow \frac{A}{t} = \frac{1}{2} r^2 \frac{\theta}{t} = \frac{1}{2} r^2 \omega$$

Now, to see the connection of this quantity with angular momentum, let us write the equation of angular momentum :

$$L = rXp_{\perp} = rmv_{\perp} = rmX\omega r = m\omega r^2$$

$$\Rightarrow \omega r^2 = \frac{L}{m}$$

Substituting this expression in the equation of area – speed, we have :

$$\Rightarrow \frac{A}{\theta} = \frac{L}{2m}$$

As no external torque is assumed to exist on the "Sun-planet" system, its angular momentum is conserved. Hence, parameters on the right hand side of the equation i.e. "L" and "m" are constants. This yields that area-speed is constant as proposed by Kepler.

We can interpret the above result other way round also. Kepler's law says that area-speed of a planet is constant. His observation is based on measured data by Tycho Brahe. It implies that angular momentum of the system remains constant. This means that no external torque applies on the "Sun – planet" system.

16.3.4.3 Law of time periods

The law relates time period of the planet with semi major axis of the elliptical trajectory.

Definition 16.3: Law of time periods
The square of the time period of a planet is proportional to the cube of the semi-major axis.

We have already seen in the case of circular trajectory around a larger mass and also in the case of two body system (see Two body system - circular motion)in which each body is moving along two circular trajectories that time period is given by :

$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{(GM)}}$$

The expression of time period for elliptical trajectory is similar except that semi-major axis replaces "r". We have not proved this in the module, but can be so derived. Squaring each of the side and replacing "r" by "a", we have :

$$\Rightarrow T^2 = \frac{4\pi^2 a^3}{GM}$$

$$\Rightarrow T^2 \propto a^3$$

16.4 Conclusions

Thus, we conclude the following :
1: The planet follows a elliptical path about Sun.
2: The Sun lies at one of the foci.
3: Gravitational force, centripetal force, linear and angular velocities are variable with the motion.
4: Velocities are maximum at perihelion and minimum at aphelion.
5: Although angular velocity is variable, the angular momentum of the system is conserved.
6: The expression of total mechanical energy is same as in the case of circular motion with the exception that semi major axis, "a", replaces radius, "r".
7: The expression of time period is same as in the case of circular motion with the exception that semi major axis, "a", replaces radius, "r".
8: Kepler's three laws are consistent with Newtonian mechanics.

16.5 Acknowldegement

Author wishes to place special thanks to Mr. Mark Prange, Port Isabel, Texas, USA and David F Shively for their valuable editorial suggestions on the subject discussed in this module.

Glossary

B

Binding energy

Binding energy is equal to the modulus of mechanical energy.

G

Gravitational potential energy

The gravitational potential energy of a system of particles is equal to the work by the external force as a particle is brought from infinity slowly to its position in the presence of other particles of the system.

Gravitational potential energy

The gravitational potential energy of a system of particles is equal to "negative" of the work by the gravitational force as a particle is brought from infinity to its position in the presence of other particles of the system.

Gravitational potential

The gravitational potential at a point is equal to the work by the external force as a particle of unit mass is brought from infinity to its position in the gravitational field.

Gravitational potential

The gravitational potential at a point is equal to "negative" of the work by the gravitational force as a particle of unit mass is brought from infinity to its position in the gravitational field.

L

Law of orbits

The orbit of every planet is an ellipse with the sun at one of the foci.

Law of time periods

The square of the time period of a planet is proportional to the cube of the semi-major axis.

Law of velocities

The line joining a planet and Sun sweeps equal area in equal times in the planet's orbit.

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