Radicals

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CONNEXIONS

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Chapter 1

Radical Concepts – Introduction[1]

The concept of a radical (or root) is a familiar one, and was reviewed in the conceptual explanation of logarithms in the previous chapter. In this chapter, we are going to explore some possibly unfamiliar properties of radicals, and solve equations involving radicals.

[1]This content is available online at <http://cnx.org/content/m18244/1.3/>.

Chapter 2

Radical Concepts – Properties of Radicals[1]

What is $\sqrt{x^2+9}$? Many students will answer quickly that the answer is $(x+3)$ and have a very difficult time believing this answer is wrong. But it is wrong.

$\sqrt{x^2}$ **is** $x^{*\ 2}$ and $\sqrt{9}$ **is** 3, but $\sqrt{x^2+9}$ **is not** $(x+3)$.

Why not? Remember that $\sqrt{x^2+9}$ is asking a question: "what **squared** gives the answer x^2+9 ?" So $(x+3)$ is not an answer, because $(x+3)^2 = x^2 + 6\text{x}{+}9$, **not** x^2+9 .

As an example, suppose $x=4$. So $\sqrt{x^2+9} = \sqrt{4^2+9} = \sqrt{25} = 5$. But $(x+3) = 7$.

NOTE: If two numbers are **added** or **subtracted** under a square root, you cannot split them up. In symbols: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ or, to put it another way, $\sqrt{x^2+y^2} \neq a+b$

$\sqrt{x^2+9}$ cannot, in fact, be simplified at all. It is a perfectly valid function, but cannot be rewritten in a simpler form.

How about $\sqrt{9x^2}$? By analogy to the previous discussion, you might expect that this cannot be simplified either. But in fact, it can be simplified:

$\sqrt{9x^2} = 3x$

Why? Again, $\sqrt{9x^2}$ is asking "what **squared** gives the answer $9x^2$?" The answer is $3x$ because $(3x)^2 = 9x^2$.

Similarly, $\sqrt{\frac{9}{x^2}} = \frac{3}{x}$, because $\left(\frac{3}{x}\right)^2 = \frac{9}{x^2}$.

NOTE: If two numbers are multiplied or divided under a square root, you **can** split them up. In symbols: $\sqrt{\text{ab}} = \sqrt{a}\sqrt{b}$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

[1] This content is available online at <http://cnx.org/content/m18271/1.1/>.

[2] I'm fudging a bit here: $\sqrt{x^2}$ is x only if you ignore negative numbers. For instance, if $x = --3$, then $x^2 = 9$, and $\sqrt{x^2}$ is 3; so in that case, $\sqrt{x^2}$ is not x. In general, $\sqrt{x^2} = |x|$. However, this subtlety is not relevant to the overall point, which is that you cannot break up two terms that are added under a radical.

Chapter 3

Radical Concepts – Simplifying Radicals[1]

3.1 Simplifying Radicals

The property $\sqrt{ab}= \sqrt{a}\sqrt{b}$ can be used to simplify radicals. The key is to break the number inside the root into two factors, **one of which is a perfect square**.

Example 3.1: Simplifying a Radical

$\sqrt{\mathbf{75}}$	
$=\sqrt{25\bullet 3}$	**because 25•3 is 75, and 25 is a perfect square**
$=\sqrt{25}\sqrt{3}$	**because** $\sqrt{\mathbf{ab}} = \sqrt{a} \quad \sqrt{b}$
$=5 \quad \sqrt{3}$	**because** $\sqrt{25}=\mathbf{5}$

Table 3.1

So we conclude that $\sqrt{75}=5\ \sqrt{3}$. You can confirm this on your calculator (both are approximately 8.66).

We rewrote 75 as $25 \bullet 3$ because 25 is a perfect square. We could, of course, also rewrite 75 as $5 \bullet 15$, but—although correct—that would not help us simplify, because neither number is a perfect square.

Example 3.2: Simplifying a Radical in Two Steps

$\sqrt{180}$	
$=\sqrt{9\bullet 20}$	**because** $9\bullet 20$ **is 180, and 9 is a perfect square**
$=\sqrt{9}\ \sqrt{20}$	**because** $\sqrt{\mathbf{ab}} = \sqrt{a} \quad \sqrt{b}$
$=3\ \sqrt{20}$	**So far, so good. But wait! We're not done!**
$=3\ \sqrt{4\bullet 5}$	**There's another perfect square to pull out!**
$=3\ \sqrt{4}\sqrt{5}$	
$=3\,(2)\ \sqrt{5}$	
$=6\ \sqrt{5}$	**Now we're done.**

[1]This content is available online at <http://cnx.org/content/m18274/1.3/>.

Table 3.2

The moral of this second example is that **after** you simplify, you should always look to see if you can simplify **again** .

A secondary moral is, try to pull out the biggest perfect square you can. We could have jumped straight to the answer if we had begun by rewriting 180 as 36 • 5.

This sort of simplification can sometimes allow you to **combine** radical terms, as in this example:

Example 3.3: Combining Radicals

$\sqrt{\mathbf{75}}$ −− $\sqrt{\mathbf{12}}$	
$= 5\ \sqrt{3}\ - -2\ \sqrt{3}$	**We found earlier that $\sqrt{\mathbf{75}}$ $=5$ $\sqrt{3}$. Use the same method to confirm that $\sqrt{\mathbf{12}}$ $=2$ $\sqrt{3}$.**
$= 3\ \sqrt{3}$	**5 of anything minus 2 of that same thing is 3 of it, right?**

Table 3.3

That last step may take a bit of thought. It can only be used when the radical is the same. Hence, $\sqrt{2}$ + $\sqrt{3}$ cannot be simplified at all. We were able to simplify $\sqrt{\mathbf{75}}$ − $\sqrt{\mathbf{12}}$ only by making the radical in both cases the same .

So why does $5\ \sqrt{3} - -2\ \sqrt{3} = 3\ \sqrt{3}$? It may be simplest to think about verbally: 5 of these things, minus 2 of the same things, is 3 of them. But you can look at it more formally as a factoring problem, if you see a common factor of $\sqrt{3}$.

$5\ \sqrt{3} - -2\ \sqrt{3} = \sqrt{3}(5 - -2) = \sqrt{3}(3)$.

Of course, the process is exactly the same if variable are involved instead of just numbers!

Example 3.4: Combining Radicals with Variables

$x^{\frac{3}{2}} + x^{\frac{5}{2}}$	
$= x^3 + x^5$	**Remember the definition of fractional exponents!**
$= \sqrt{x^2 * x} + \sqrt{x^4 * x}$	**As always, we simplify radicals by factoring them inside the root...**
$\sqrt{x^2} * \sqrt{x} + \sqrt{x^4} * \sqrt{x}$	**and then breaking them up...**
$= x\sqrt{x} + x^2\sqrt{x}$	**and then taking square roots outside!**
$= \left(x^2 + x\right)\sqrt{x}$	**Now that the radical is the same, we can combine.**

Table 3.4

3.1.1 Rationalizing the Denominator

It is always possible to express a fraction with no square roots in the denominator.

Is it always desirable? Some texts are religious about this point: "You should never have a square root in the denominator." I have absolutely no idea why. To me, $\frac{1}{\sqrt{2}}$ looks simpler than $\frac{\sqrt{2}}{2}$; I see no overwhelming reason for forbidding the first or preferring the second.

However, there are times when it is useful to remove the radicals from the denominator: for instance, when adding fractions. The trick for doing this is based on the basic rule of fractions: **if you multiply the top and bottom of a fraction by the same number, the fraction is unchanged.** This rule enables us to say, for instance, that $\frac{2}{3}$ is exactly the same number as $\frac{2\cdot3}{3\cdot3}=\frac{6}{9}$.

In a case like $\frac{1}{\sqrt{2}}$, therefore, you can multiply the top and bottom by $\sqrt{2}$.

$\frac{1}{\sqrt{2}}=\frac{1*2}{\sqrt{2}*\sqrt{2}}=\frac{\sqrt{2}}{2}$

What about a more complicated case, such as $\frac{\sqrt{12}}{1+\sqrt{3}}$? You might think we could simplify this by multiplying the top and bottom by $(1+\sqrt{3})$, but that doesn't work: the bottom turns into $(1+3)^2=1+2\sqrt{3}+3$, which is at least as ugly as what we had before.

The correct trick for getting rid of $(1+\sqrt{3})$ is to multiply it by $(1--\sqrt{3})$. These two expressions, identical except for the replacement of a+ by a−, are known as **conjugates**. What happens when we multiply them? We don't need to use FOIL if we remember that

$(x+y)(x-y)=x^2-y^2$

Using this formula, we see that

$\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)=1^2-\left(\sqrt{3}\right)^2=1-3=-2$

So the square root does indeed go away. We can use this to simplify the original expression as follows.

Example 3.5: Rationalizing Using the Conjugate of the Denominator

$\frac{\sqrt{12}}{1+\sqrt{3}}=\frac{\sqrt{12}\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}=\frac{\sqrt{12}-\sqrt{36}}{1-3}=\frac{2\sqrt{3}-6}{-2}=-\sqrt{3}+3$

As always, you may want to check this on your calculator. Both the original and the simplified expression are approximately 1.268.

Of course, the process is the same when variables are involved.

Example 3.6: Rationalizing with Variables

$\frac{1}{x-\sqrt{x}}=\frac{1\left(x+\sqrt{x}\right)}{\left(x-\sqrt{x}\right)\left(x+\sqrt{x}\right)}=\frac{x+\sqrt{x}}{x^2-x}$

Once again, we multiplied the top and the bottom by the **conjugate of the denominator**: that is, we replaced a− with a+. The formula $(x+a)(x-a)=x^2-a^2$ enabled us to quickly multiply the terms on the bottom, and eliminated the square roots in the denominator.

Chapter 4

Radical Concepts – Radical Equations[1]

When solving **equations that involve radicals**, begin by asking yourself: **is there an x under the square root?** The answer to this question will determine the way you approach the problem.

If there is **not** an x under the square root—if only numbers are under the radicals—you can solve much the same way you would solve with no radicals at all.

Example 4.1: Radical Equation with No Variables Under Square Roots

$\sqrt{2}x+5=7--\sqrt{3}x$	Sample problem: no variables under radicals
$\sqrt{2}+\sqrt{3}x=7-5$	Get everything with an x on one side, everything else on the other
$x\left(\sqrt{2}+\sqrt{3}\right)=2$	Factor out the x
$x=\frac{2}{\sqrt{2}+\sqrt{3}}$	Divide, to solve for x

Table 4.1

The key thing to note about such problems is that **you do not have to square both sides of the equation**. $\sqrt{2}$ may look ugly, but it is just a number—you could find it on your calculator if you wanted to—it functions in the equation just the way that the number 10, or $\frac{1}{3}$, or π would.

If there is an x under the square root, the problem is completely different. You will have to square both sides to get rid of the radical. However, there are two important notes about this kind of problem.

1. Always get the radical **alone, on one side of the equation**, before squaring.
2. Squaring both sides can introduce **false answers**—so it is important to check your answers after solving!

Both of these principles are demonstrated in the following example.

Example 4.2: Radical Equation with Variables under Square Roots

[1]This content is available online at <http://cnx.org/content/m18273/1.3/>.

$\sqrt{x+2}+3x = 5x + 1$	Sample problem with variables under radicals
$\sqrt{x+2}=2x+1$	Isolate the radical before squaring!
$x+2=(2x+1)^2$	Now, square both sides
$x+2=4x^2+4x+1$	Multiply out. Hey, it looks like a quadratic equation now!
$x+2=4x^2+4x+1$	As always with quadratics, get everything on one side.
$(4x-1)(x+1)=0$	Factoring: the easiest way to solve quadratic equations.
$x=\frac{1}{4}$ or $x=-1$	Two solutions**. Do they work?** Check in the **original equation**!

Table 4.2

Check $x=\frac{1}{4}$	**Check** $x=--1$
$\sqrt{\frac{1}{4}+2}+3\left(\frac{1}{4}\right)\overset{?}{=}5\left(\frac{1}{4}\right)+1$	$\sqrt{-1+2}+3(-1)\overset{?}{=}5(-1)+1$
$\sqrt{\frac{1}{4}+\frac{8}{4}}+\frac{3}{4}\overset{?}{=}\frac{5}{4}+1$	$\sqrt{1}-3\overset{?}{=}-5+1$
$\sqrt{\frac{9}{4}}+\frac{3}{4}\overset{?}{=}\frac{5}{4}+\frac{4}{4}$	$1-3\overset{?}{=}-5+1$
$\frac{3}{2}+\frac{3}{4}\overset{?}{=}\frac{5}{4}+\frac{4}{4}$	$-2=-4$ **Not equal!**
$\frac{9}{4}=\frac{9}{4}$	

Table 4.3

So the algebra yielded two solutions: $\frac{1}{4}$ and –1. Checking, however, we discover that only the first solution is valid. This problem demonstrates how important it is to check solutions whenever squaring both sides of an equation.

If variables under the radical occur more than once, you will have to go through this procedure multiple times. Each time, you isolate a radical and then square both sides.

Example 4.3: Radical Equation with Variables under Square Roots Multiple Times

$\sqrt{x+7}-x=1$	Sample problem with variables under radicals multiple times
$\sqrt{x+7}=\sqrt{x}+1$	Isolate one radical. (I usually prefer to start with the bigger one.)
$x+7=x+2\sqrt{x}+1$	Square both sides. The **two**-radical equation is now a **one**-radical equation.
$6=2\sqrt{x}$	
$3=x$	Isolate the remaining radical, then square both sides again..
$9=x$	In this case, we end up with only one solution. But we still need to check it.

Table 4.4

Check x=9

$\sqrt{9+7}-\sqrt{9}\overset{?}{=}1$
$\sqrt{16}-\sqrt{9}\overset{?}{=}1$
$4-3=1$

Table 4.5

Remember, the key to this problem was recognizing that **variables under the radical** occurred in the original problem **two times**. That cued us that we would have to go through the process—isolate a radical, then square both sides—twice, before we could solve for x. And **whenever** you square both sides of the equation, it's vital to check your answer(s)!

4.1 When good math leads to bad answers

Why is it that—when squaring both sides of an equation—perfectly good algebra can lead to invalid solutions? The answer is in the redundancy of squaring. Consider the following equation:

$- - 5 = 5$ False. But square both sides, and we get...

$25 = 25$ True. So squaring both sides of a **false** equation can produce a **true** equation.

To see how this affects our equations, try plugging $x = -1$ into the various steps of the first example.

Example 4.4: Why did we get a false answer of x=–1 in Example 1?

$\sqrt{x+2} + 3\text{x} = 5\text{x} + 1$	Does $x = -1$ work here? No, it does not.
$\sqrt{x+2} = 2\text{x} + 1$	How about here? No, $x = -1$ produces the false equation 1=–1.
$x + 2 = (2\text{x} + 1)^2$	Suddenly, $x = -1$ works. (Try it!)

Table 4.6

When we squared both sides, we "lost" the difference between 1 and –1, and they "became equal."

From here on, when we solved, we ended up with $x = -1$ as a valid solution.

Test your memory: When you square both sides of an equation, you can introduce false answers. We have encountered one other situation where **good algebra** can lead to a **bad answer**. When was it?

Answer: It was during the study of absolute value equations, such as $|2\text{x} + 3| = -11\text{x} + 42$. In those equations, we also found the hard-and-fast rule that you **must check your answers** as the last step.

What do these two types of problem have in common? The function $|x|$ actually has a lot in common with x^2. Both of them have the peculiar property that they always turn $-$a and a into the same response. (For instance, if you plug –3 and 3 into the function, you get the same thing back.) This property is known as being an **even function**. Dealing with such "redundant" functions leads, in both cases, to the possibility of false answers.

The similarity between these two functions can also be seen in the graphs: although certainly not identical, they bear a striking resemblance to each other. In particular, both graphs are symmetric about the y-axis, which is the fingerprint of an "even function".

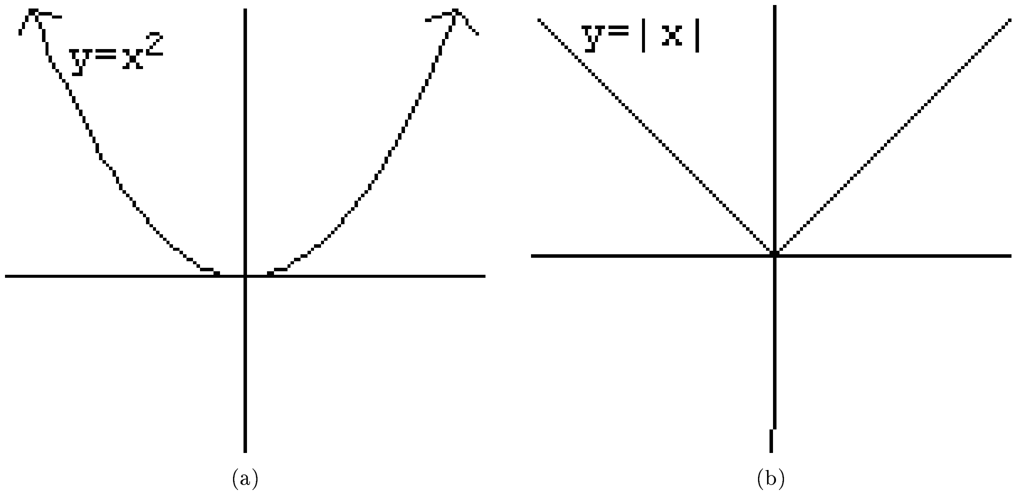

Figure 4.1

Chapter 5

Radicals Homework – Radical Equations[1]

Before I get into the radical equations, there is something very important I have to get out of the way. Square these two out:

Exercise 5.1
$\left(2+\sqrt{2}\right)^2 =$

Exercise 5.2
$\left(\sqrt{3}+\sqrt{2}\right)^2 =$

How'd it go? If you got six for the first answer and five for the second, **stop!** Go back and look again, because those answers are not right. (If you don't believe me, try it on your calculator.) When you've got those correctly simplified (feel free to ask—or, again, check on your calculator) then go on.

Now, radical equations. Let's start off with **an easy radical equation.**

Exercise 5.3
$\sqrt{2}x+3=7$

I call this an "easy" radical equation because there is no x under the square root. Sure, there's a , but that's just a number. So you can solve it pretty much the same way you would solve $4x+3=7$; just subtract 3, then divide by $\sqrt{2}$.

a. Solve for x
b. Check your answer by plugging it into the original equation. Does it work?

This next one is definitely trickier, but it is still in the category that I call "easy" because there is still no x under the square root.

Exercise 5.4
$\sqrt{2}x+3x=7$

a. Solve for x
b. Check your answer by plugging it into the original equation. Does it work? (Feel free to use your calculator, but show me what you did and how it came out.)

Now, what if there **is** an x under the square root? Let's try a basic one like that.

Exercise 5.5
Solve for x: $\sqrt{x}=9$

What did you get? If you said the answer is three: shame, shame. The square root of 3 isn't 9, is it? Try again.

[1] This content is available online at <http://cnx.org/content/m19272/1.1/>.

OK, that's better. You probably guessed your way to the answer. But if you had to be systematic about it, you could say "I got to the answer by squaring both sides." The rule is: **whenever there is an x under a radical, you will have to square both sides. If there is no x under the radical, don't square both sides.**

It worked out this time, but squaring both sides is fraught with peril. Here are a few examples.

Exercise 5.6
$\sqrt{x} = -9$

a. Solve for x, by squaring both sides.
b. Check your answer by plugging it into the original equation.

Hey, what happened? When you square both sides, you get $x = 81$, just like before. But this time, it's the wrong answer: $\sqrt{81}$ is not -9. **The moral of the story is that when you square both sides, you can introduce false answers.** So whenever you square both sides, you have to check your answers to see if they work. (We will see that rule come up again in some much less obvious places, so it's a good idea to get it under your belt now: **whenever you square both sides, you can introduce false answers!**)

But that isn't the only danger of squaring both sides. Check this out...

Exercise 5.7
Solve for x by squaring both sides: $2 + \sqrt{x} = 5$

Hey, what happened there? When you square the left side, you got (I hope) $x + 4\sqrt{x} + 4$. Life isn't any simpler, is it? So the lesson there is, **you have to get the square root by itself before you can square both sides**. Let's come back to that problem.

Exercise 5.8
$2 + \sqrt{x} = 5$

a. Solve for x by first getting the square root by itself, and then squaring both sides
b. Check your answer in the original equation.

Whew! Much better! Some of you may have never fallen into the trap—you may have just subtracted the two to begin with. But you will find you need the same technique for harder problems, such as this one:

Exercise 5.9
$x - \sqrt{x} = 6$

a. Solve for x by first getting the square root by itself, and then squaring both sides, and then solving the resulting equation.

NOTE: You should end up with two answers.

b. Check your answers in the original equation.

NOTE: If you did everything right, you should find that one answer works and the other doesn't. Once again, we see that squaring both sides can introduce false answers!

Exercise 5.10
$\sqrt{x-2} = \sqrt{3} - \sqrt{x}$

What do you do now? You're going to have to square both sides...that will simplify the left, but the right will still be ugly. But if you look closely, you will see that you have changed an equation with x under the square root twice, into an equation with x under the square root once. So then, you can solve it the way you did above: get the square root by itself and square both sides. Before you are done, you will have squared both sides twice!

Solve #10 and check your answers...

Index of Keywords and Terms

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