Signals, Systems, and Transforms (under construction)

Collection Editor:

Richard Baraniuk

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Collection Editor:

Richard Baraniuk

Authors:

Richard Baraniuk
Catherine Elder
Michael Haag
Stephen Kruzick
Mariyah Poonawala
Ricardo Radaelli-Sanchez
Justin Romberg
Melissa Selik
Prashant Singh

Online:
< http://cnx.org/content/col10207/1.4/ >

C O N N E X I O N S

Rice University, Houston, Texas

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Preface: Signals and Systems[1]

In this course, we will learn about signals and systems for processing signals. We will rely heavily on ideas from linear algebra and Hilbert space to unify our treatment of the four fundamental classes of signals - the four combinations of discrete-time vs. continuous-time and periodic/finite vs. aperiodic/infinite.

In contrast to most textbooks, we begin with the discrete Fourier transform for discrete-time, periodic/finite signals. Hence, this course could be subtitled: **DFT First**. For a more standard treatment starting with continuous-time signals, see the Connexions course Signals and Systems[2] which was used until 2002 and in 2004.

This course also marks the introduction of National Instruments **Labview VIs**[3] to the Connexions system. Look for them in select modules - they are designed to help students visualize important concepts.

Comments, typos, and suggestions are welcome.

[1]This content is available online at <http://cnx.org/content/m11483/1.6/>.
[2]http://cnx.rice.edu/content/col10064/latest/
[3]http://ni.com/labview

Chapter 1

Introduction to Signals, Systems, and Transforms

1.1 Signals[1]

1.2 Operators[2]

1.2.1 Operators

1.2.2 Systems

systems "process"/change

1.2.3 Transforms

"re-express/translate"

1.3 System Classifications and Properties[3]

1.3.1 Introduction

In this module some of the basic classifications of systems will be briefly introduced and the most important properties of these systems are explained. As can be seen, the properties of a system provide an easy way to distinguish one system from another. Understanding these basic differences between systems, and their properties, will be a fundamental concept used in all signal and system courses. Once a set of systems can be identified as sharing particular properties, one no longer has to reprove a certain characteristic of a system each time, but it can simply be known due to the the system classification.

1.3.2 Classification of Systems

1.3.2.1 Continuous vs. Discrete

One of the most important distinctions to understand is the difference between discrete time and continuous time systems. A system in which the input signal and output signal both have continuous domains is said to

[1]This content is available online at <http://cnx.org/content/m11501/1.2/>.
[2]This content is available online at <http://cnx.org/content/m11499/1.4/>.
[3]This content is available online at <http://cnx.org/content/m10084/2.21/>.

be a continuous system. One in which the input signal and output signal both have discrete domains is said to be a continuous system. Of course, it is possible to conceive of signals that belong to neither category, such as systems in which sampling of a continuous time signal or reconstruction from a discrete time signal take place.

1.3.2.2 Linear vs. Nonlinear

A linear system is any system that obeys the properties of scaling (first order homogeneity) and superposition (additivity) further described below. A nonlinear system is any system that does not have at least one of these properties.

To show that a system H obeys the scaling property is to show that

$$H\left(kf\left(t\right)\right)=kH\left(f\left(t\right)\right) \tag{1.1}$$

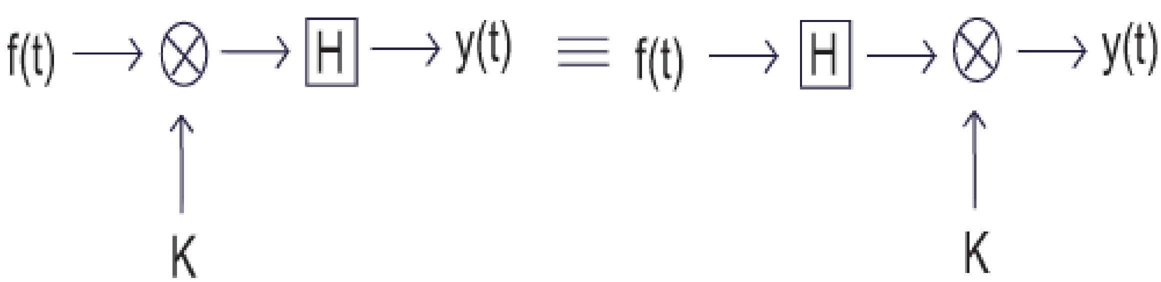


Figure 1.1: A block diagram demonstrating the scaling property of linearity

To demonstrate that a system H obeys the superposition property of linearity is to show that

$$H\left(f_1\left(t\right)+f_2\left(t\right)\right)=H\left(f_1\left(t\right)\right)+H\left(f_2\left(t\right)\right) \tag{1.2}$$

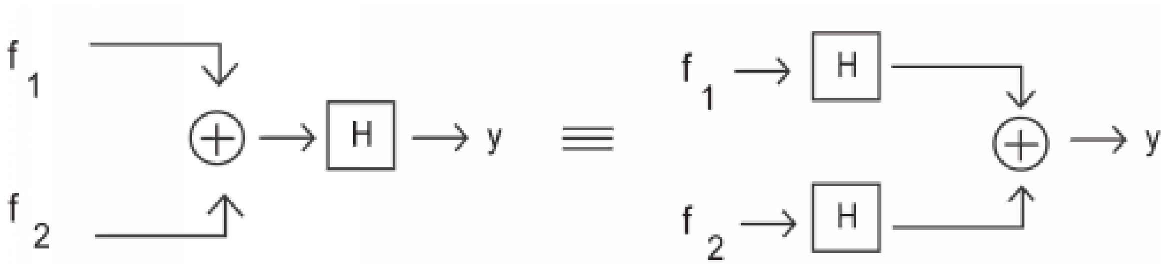


Figure 1.2: A block diagram demonstrating the superposition property of linearity

It is possible to check a system for linearity in a single (though larger) step. To do this, simply combine the first two steps to get

$$H\left(k_1 f_1\left(t\right) + k_2 f_2\left(t\right)\right) = k_2 H\left(f_1\left(t\right)\right) + k_2 H\left(f_2\left(t\right)\right) \tag{1.3}$$

1.3.2.3 Time Invariant vs. Time Varying

A system is said to be time invariant if it commutes with the parameter shift operator defined by $S_T\left(f\left(t\right)\right) = f\left(t - T\right)$ for all T, which is to say

$$HS_T = S_T H \tag{1.4}$$

for all real T. Intuitively, that means that for any input function that produces some output function, any time shift of that input function will produce an output function identical in every way except that it is shifted by the same amount. Any system that does not have this property is said to be time varying.

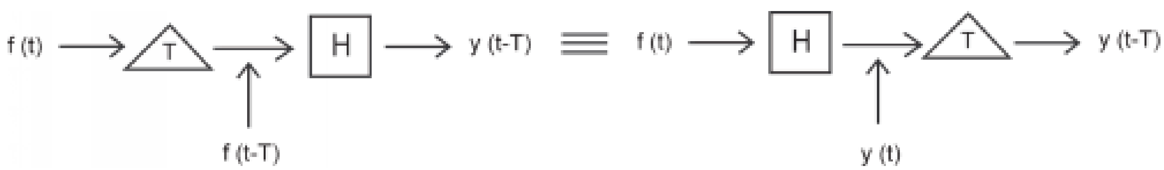


Figure 1.3: This block diagram shows what the condition for time invariance. The output is the same whether the delay is put on the input or the output.

1.3.2.4 Causal vs. Noncausal

A causal system is one in which the output depends only on current or past inputs, but not future inputs. Similarly, an anticausal system is one in which the output depends only on current or future inputs, but not past inputs. Finally, a noncausal system is one in which the output depends on both past and future inputs. All "realtime" systems must be causal, since they can not have future inputs available to them.

One may think the idea of future inputs does not seem to make much physical sense; however, we have only been dealing with time as our dependent variable so far, which is not always the case. Imagine rather that we wanted to do image processing. Then the dependent variable might represent pixel positions to the left and right (the "future") of the current position on the image, and we would not necessarily have a causal system.

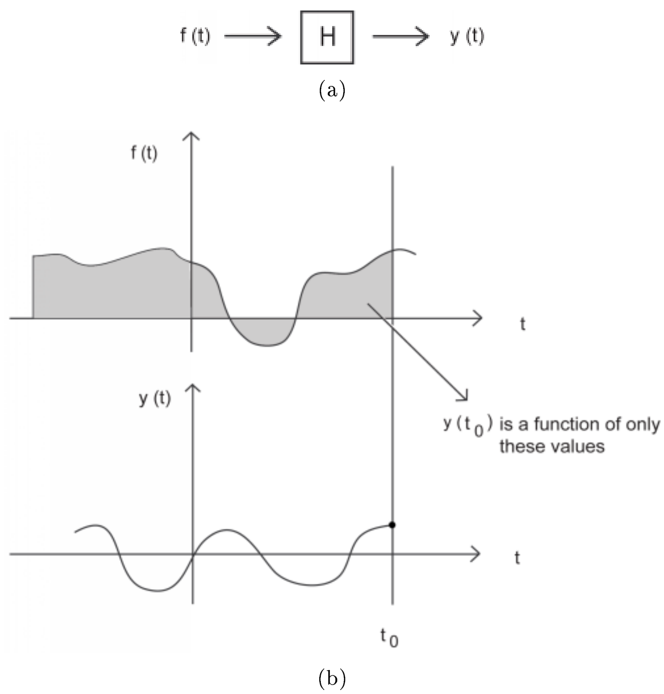


Figure 1.4: (a) For a typical system to be causal... (b) ...the output at time t_0, $y(t_0)$, can only depend on the portion of the input signal before t_0.

1.3.2.5 Stable vs. Unstable

There are several definitions of stability, but the one that will be used most frequently in this course will be bounded input, bounded output (BIBO) stability. In this context, a stable system is one in which the output is bounded if the input is also bounded. Similarly, an unstable system is one in which at least one bounded input produces an unbounded output.

Representing this mathematically, a stable system must have the following property, where $x(t)$ is the input and $y(t)$ is the output. The output must satisfy the condition

$$|y(t)| \leq M_y < \infty \tag{1.5}$$

whenever we have an input to the system that satisfies

$$|x(t)| \leq M_x < \infty \tag{1.6}$$

M_x and M_y both represent a set of finite positive numbers and these relationships hold for all of t. Otherwise, the system is unstable.

1.3.3 System Classifications Summary

This module describes just some of the many ways in which systems can be classified. Systems can be continuous time, discrete time, or neither. They can be linear or nonlinear, time invariant or time varying, and stable or unstable. We can also divide them based on their causality properties. There are other ways to classify systems, such as use of memory, that are not discussed here but will be described in subsequent modules.

1.4 Transforms[4]

Add links examples such as Laplace, fourier and wavelets

[4]This content is available online at <http://cnx.org/content/m11500/1.2/>.

Chapter 2

Signals

2.1 Signal Basics

2.1.1 Signals are functions[1]

A signal is a function that maps an independent variable into a dependent variable. The function f(x), for each value of x, produces the value f(x)

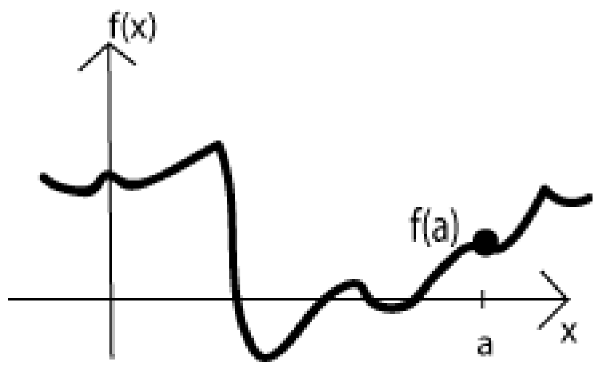


Figure 2.1

There are four ways to classify signals according to the values that the independent and dependent variables can take. Refer to the table below.

[1]This content is available online at <http://cnx.org/content/m11502/1.4/>.

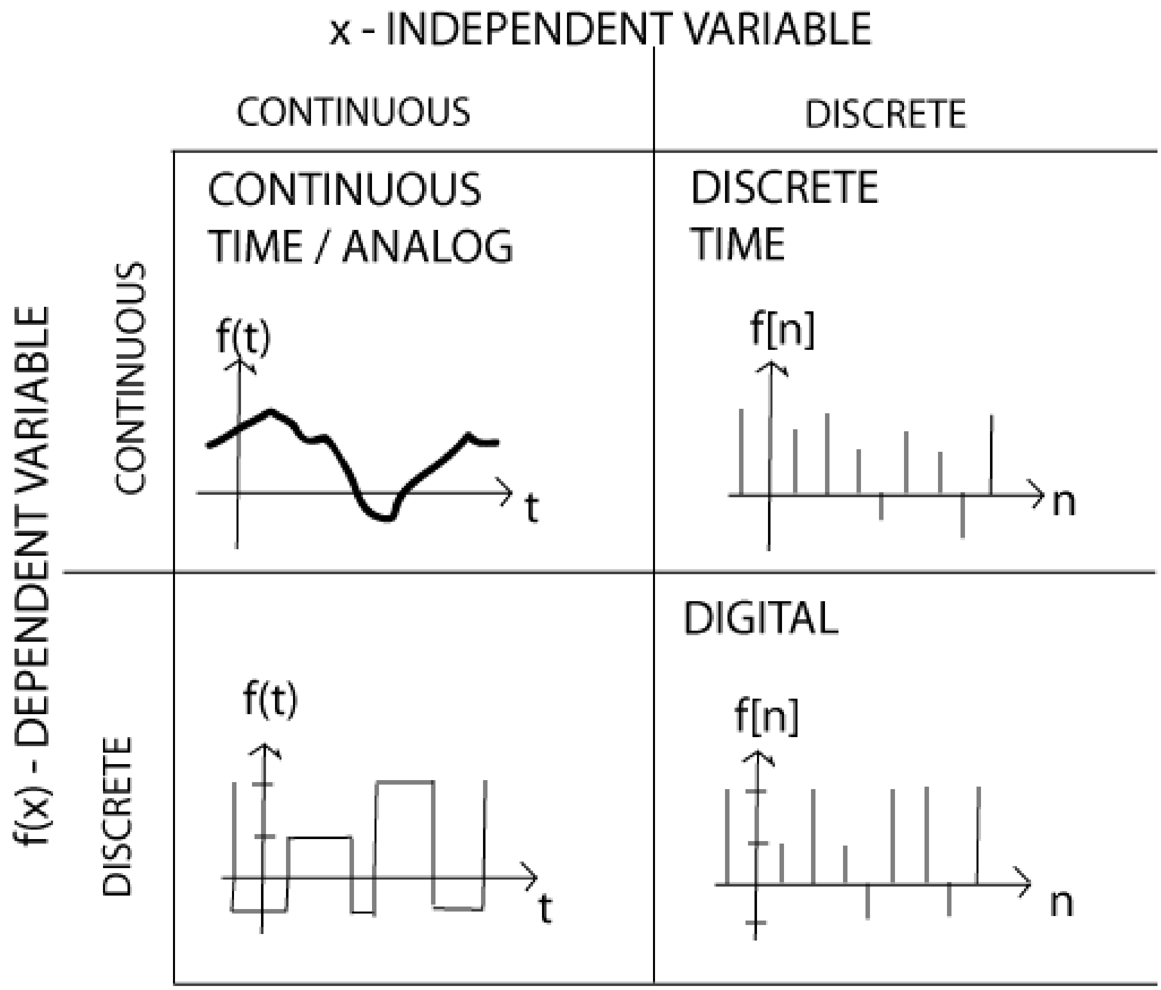


Figure 2.2

2.1.1.1 Quick Aside on Signal Notation

Continuous Time signals are represented as $f(t)$ where $t \in \mathbb{R}$. Discrete Time signals are represented as $f[n]$ where $n \in \mathbb{Z}$

2.1.2 The Four Fundamental Types of Signals[2]

2.1.2.1 Continuous-Time, Finite Length Signals

2.1.2.2 Continuous-Time, Infinite Length Signals

2.1.2.3 Discrete-Time, Finite Length Signals

2.1.2.4 Discrete-Time, Infinite Length Signals

2.1.3 Introduction to Sampling and Reconstruction[3]

Potential Existing modules: Sampling[4] Reconstruction[5]

2.2 Properties of Signals

2.2.1 Analog and Digital Signals[6]

2.2.2 Continuous Time Periodic Signals[7]

2.2.2.1 Introduction

This module describes the type of signals acted on by the Continuous Time Fourier Series.

2.2.2.2 Relevant Spaces

The Continuous-Time Fourier Series maps finite-length (or T-periodic), continuous-time signals in L^2 to infinite-length, discrete-frequency signals in l^2.

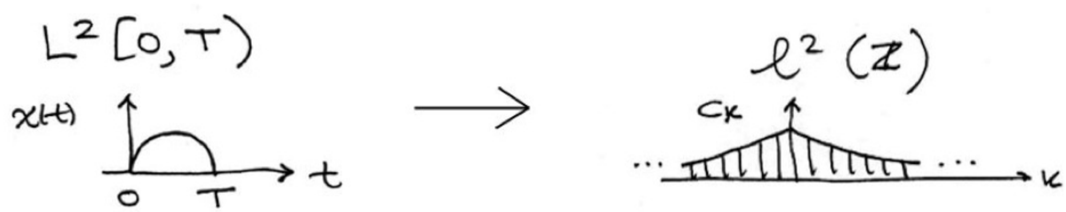


Figure 1: Mapping $L^2([0, T))$ in the time domain to $l^2(\mathbb{Z})$ in the frequency domain.

2.2.2.3 Periodic Signals

When a function repeats itself exactly after some given period, or cycle, we say it's **periodic**. A **periodic function** can be mathematically defined as:

$$f(t) = f(t + mT)\,\forall m : (m \in \mathbb{Z}) \tag{2.1}$$

where $T > 0$ represents the **fundamental period** of the signal, which is the smallest positive value of T for the signal to repeat. Because of this, you may also see a signal referred to as a T-periodic signal. Any function that satisfies this equation is said to be **periodic** with period T.

We can think of **periodic functions** (with period T) two different ways:

[2]This content is available online at <http://cnx.org/content/m11503/1.1/>.
[3]This content is available online at <http://cnx.org/content/m11530/1.1/>.
[4]"Signal Sampling" <http://cnx.org/content/m10798/latest/>
[5]"Signal Reconstruction" <http://cnx.org/content/m10788/latest/>
[6]This content is available online at <http://cnx.org/content/m11504/1.1/>.
[7]This content is available online at <http://cnx.org/content/m10744/2.13/>.

1. as functions on **all** of $\mathbb{R}$

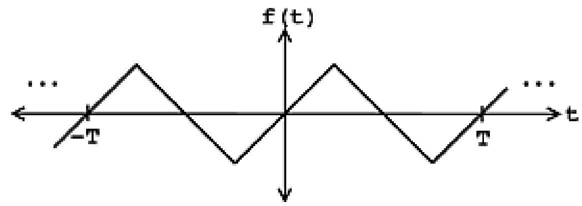


Figure 2.3: Continuous time periodic function over all of $\mathbb{R}$ where $f(t_0) = f(t_0 + T)$

2. or, we can cut out all of the redundancy, and think of them as functions on an interval $[0, T]$ (or, more generally, $[a, a + T]$). If we know the signal is T-periodic then all the information of the signal is captured by the above interval.

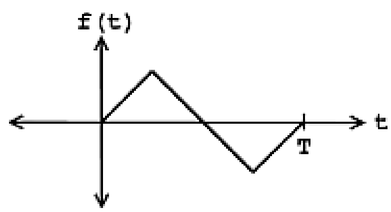


Figure 2.4: Remove the redundancy of the period function so that $f(t)$ is undefined outside $[0, T]$.

An **aperiodic** CT function $f(t)$, on the other hand, does not repeat for **any** $T \in \mathbb{R}$; *i.e.* there exists no T such that this equation (2.1) holds.

2.2.2.4 Demonstration

Here's an example demonstrating a **periodic** sinusoidal signal with various frequencies, amplitudes and phase delays:

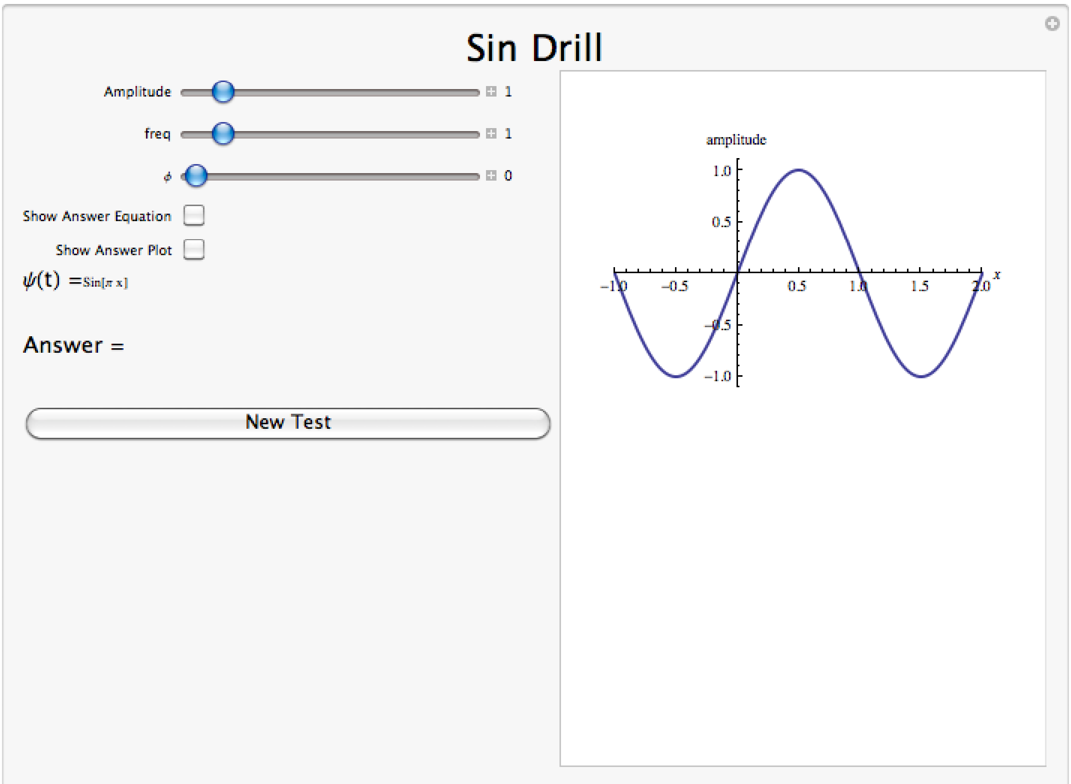


Figure 2.5: Interact (when online) with a Mathematica CDF demonstrating a Periodic Sinusoidal Signal with various frequencies, amplitudes, and phase delays. To download, right click and save file as .cdf.

To learn the full concept behind periodicity, see the video below.

Khan Lecture on Periodic Signals

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/tJW_a6JeXD8&rel=0&color1=0xb1b1b1&color2=0xd0d0d0&hl=en_US&feature=player_en

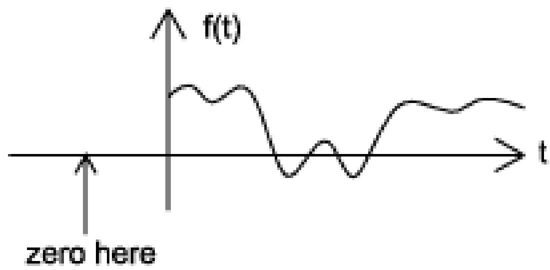
Figure 2.6: video from Khan Academy

2.2.2.5 Conclusion

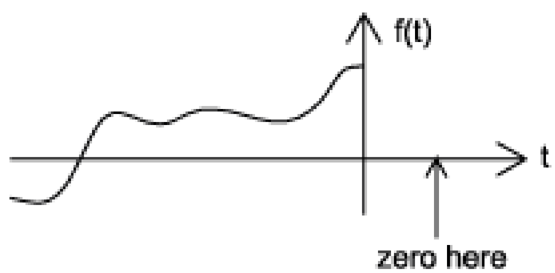
A periodic signal is completely defined by its values in one period, such as the interval [0,T].

2.2.3 Causal Signals[8]

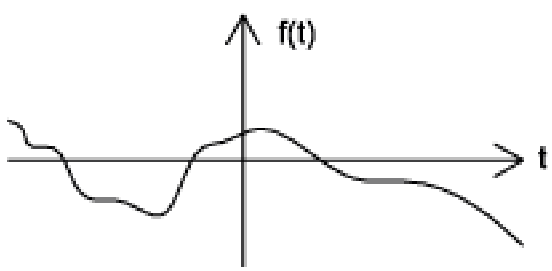
Causal signals are signals that are zero for all negative time, while **anitcausal** are signals that are zero for all positive time. **Noncausal** signals are signals that have nonzero values in both positive and negative time.



(a)



(b)



(c)

Figure 2.7: (a) A causal signal (b) An anticausal signal (c) A noncausal signal

[8]This content is available online at <http://cnx.org/content/m11495/1.3/>.

2.2.4 Real and Complex Signals[9]

A **real** signal $f(t)$ takes for each independent variable t a **real** value $f(t)$.

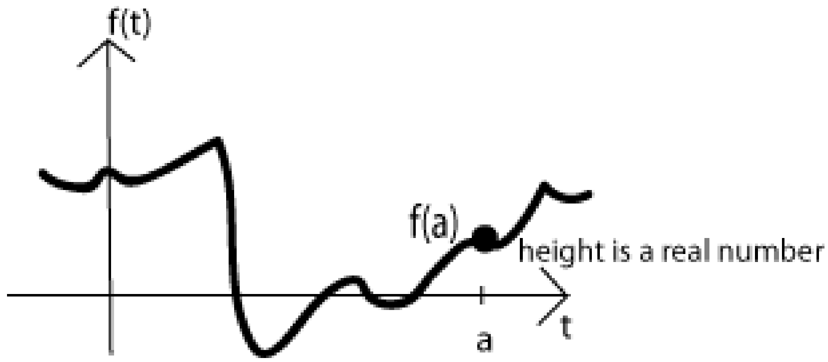


Figure 2.8

A **complex** signal $f(t)$ takes for each independent variable t a **complex** value $f(t)$.

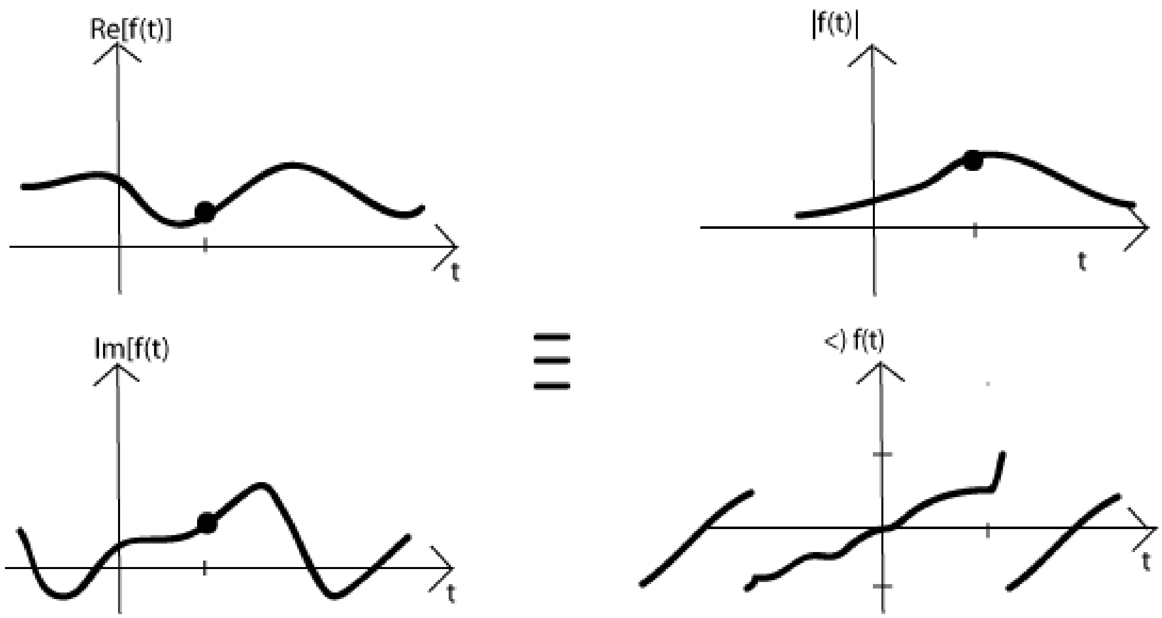


Figure 2.9

[9]This content is available online at <http://cnx.org/content/m11529/1.2/>.

2.3 Important Signals

2.3.1 Delta Function - Heuristic Definition[10]

2.3.2 Delta Function as a Generalized Function[11]

Check out module m10170 - The Impulse Function

[10] This content is available online at <http://cnx.org/content/m11485/1.1/>.
[11] This content is available online at <http://cnx.org/content/m11484/1.3/>.

2.3.3 Sifting Property of the Delta Function[12]

2.3.4 Step Function in Continuous-Time[13]

2.3.5 Step Function in Discrete Time[14]

2.3.6 Sinusoids in Continuous Time[15]

2.3.6.1 Finite Length

2.3.6.2 Infinite Length

2.3.7 Sinusoids in Discrete Time[16]

2.3.7.1 Finite Length

2.3.7.2 Infinite Length

2.3.8 Sinc Function in Continuous Time[17]

2.3.9 Sinc Function in Discrete Time[18]

2.3.10 Complex Exponential in Continuous Time[19]

2.3.11 Complex Exponential in Discrete Time[20]

2.4 Size of a signal

2.4.1 Energy of a Signal[21]

2.4.1.1 Continuous Time Finite Length"

2.4.1.2 Continuous Time Infinite Length

2.4.1.3 Discrete Time Finite Length"

2.4.1.4 Discrete Time Infinite Length

2.4.2 Norm of a Signal[22]

2.4.2.1 Continuous Time Finite Length"

2.4.2.2 Continuous Time Infinite Length

2.4.2.3 Discrete Time Finite Length"

2.4.2.4 Discrete Time Infinite Length

2.4.3 Power of a signal[23]

2.4.3.1 Continuous Time"

2.4.3.2 Discrete Time"

[12] This content is available online at <http://cnx.org/content/m11508/1.1/>.
[13] This content is available online at <http://cnx.org/content/m11486/1.2/>.
[14] This content is available online at <http://cnx.org/content/m11505/1.1/>.
[15] This content is available online at <http://cnx.org/content/m11488/1.3/>.
[16] This content is available online at <http://cnx.org/content/m11489/1.3/>.
[17] This content is available online at <http://cnx.org/content/m11506/1.1/>.
[18] This content is available online at <http://cnx.org/content/m11507/1.1/>.
[19] This content is available online at <http://cnx.org/content/m11490/1.1/>.
[20] This content is available online at <http://cnx.org/content/m11491/1.1/>.
[21] This content is available online at <http://cnx.org/content/m11509/1.1/>.
[22] This content is available online at <http://cnx.org/content/m11409/1.4/>.
[23] This content is available online at <http://cnx.org/content/m11510/1.2/>.

Chapter 3

Operators

3.1 Linearity

3.1.1 Scaling and Superposition[1]

3.1.1.1 Continuous Time

Add VI

3.1.1.2 Discrete Time

Add VI

3.1.2 Linear Operators[2]

3.1.2.1 Continuous Time

add VI

3.1.2.2 Discrete Time

add VI

3.1.3 Characterization of Linear Operators[3]

3.1.3.1 Continuous Time

link to 41

3.1.3.2 Discrete Time

Link to 41

[1]This content is available online at <http://cnx.org/content/m11512/1.1/>.
[2]This content is available online at <http://cnx.org/content/m11513/1.1/>.
[3]This content is available online at <http://cnx.org/content/m11514/1.1/>.

3.1.4 Linearity/Nonlinearity Examples[4]

3.1.4.1 Continuous Time

3.1.4.2 Discrete Time

3.2 Time Invariance

3.2.1 Time Invariant Operators[5]

3.2.2 Time Invariant/Time Variant Examples[6]

3.3 LTI Operators[7]

3.4 Characterizationi of LTI Operators[8]

3.4.1 Continuous Time

Characterization of Linear Operators (Section 3.1.3) Time Invariant Operators (Section 3.2.1)

3.4.2 Discrete Time

add VI

3.5 Discrete Time System Analysis

3.5.1 Discrete Time Signals are Vectors[9]

3.5.1.1 Finite Length Signals

The Four Fundamental types of Signals (Section 2.1.2)

3.5.1.2 Infinite Length Signals

The Four Fundamental types of Signals (Section 2.1.2)

Time Variant/Time Invariant Examples (Section 3.2.2) Step Function in Discrete Time (Section 2.3.5) Sinc Function in Discrete Time (Section 2.3.9) Sinusoids in Discrete Time (Section 2.1.2) The Complex Exponential in Discrete Time (Section 2.1.2)

3.5.2 Discrete Time Linear Systems are Matrices[10]

3.5.2.1 Finite Length

Characterization of Linear Operators (Section 3.1.3)

3.5.2.2 Infinite Length

Characterization of Linear Operators (Section 3.1.3)

[4] This content is available online at <http://cnx.org/content/m11515/1.2/>.
[5] This content is available online at <http://cnx.org/content/m11516/1.1/>.
[6] This content is available online at <http://cnx.org/content/m11517/1.1/>.
[7] This content is available online at <http://cnx.org/content/m11518/1.1/>.
[8] This content is available online at <http://cnx.org/content/m11519/1.1/>.
[9] This content is available online at <http://cnx.org/content/m11521/1.1/>.
[10] This content is available online at <http://cnx.org/content/m11522/1.1/>.

3.5.3 Discrete Time LTI systems as Matrices[11]

3.5.3.1 Finite Length

circulant Matrices

3.5.3.2 Infinite Length

Toeplitz Matrices

3.5.4 Impulse Response of a Linear Discrete Time System[12]

3.5.4.1 Finite Length

3.5.4.2 Infinite Length

Sifting Property of Delta Function (Section 2.3.3)

3.5.5 Impulse Response of an LTI Discrete Time System[13]

3.5.5.1 Finite Length

3.5.5.2 Infinite Length

3.5.6 Convolution

3.5.6.1 Discrete Time Convolution[14]

3.5.6.1.1 Introduction

Convolution, one of the most important concepts in electrical engineering, can be used to determine the output a system produces for a given input signal. It can be shown that a linear time invariant system is completely characterized by its impulse response. The sifting property of the discrete time impulse function tells us that the input signal to a system can be represented as a sum of scaled and shifted unit impulses. Thus, by linearity, it would seem reasonable to compute of the output signal as the sum of scaled and shifted unit impulse responses. That is exactly what the operation of convolution accomplishes. Hence, convolution can be used to determine a linear time invariant system's output from knowledge of the input and the impulse response.

3.5.6.1.2 Convolution and Circular Convolution

3.5.6.1.2.1 Convolution

3.5.6.1.2.1.1 Operation Definition

Discrete time convolution is an operation on two discrete time signals defined by the integral

$$(f * g)(n) = \sum_{k=-\infty}^{\infty} f(k)\, g(n-k) \tag{3.1}$$

for all signals f, g defined on $\mathbb{Z}$. It is important to note that the operation of convolution is commutative, meaning that

$$f * g = g * f \tag{3.2}$$

[11]This content is available online at <http://cnx.org/content/m11523/1.1/>.
[12]This content is available online at <http://cnx.org/content/m11524/1.1/>.
[13]This content is available online at <http://cnx.org/content/m11525/1.1/>.
[14]This content is available online at <http://cnx.org/content/m10087/2.27/>.

for all signals f, g defined on $\mathbb{Z}$. Thus, the convolution operation could have been just as easily stated using the equivalent definition

$$(f * g)(n) = \sum_{k=-\infty}^{\infty} f(n-k)\, g(k) \tag{3.3}$$

for all signals f, g defined on $\mathbb{Z}$. Convolution has several other important properties not listed here but explained and derived in a later module.

3.5.6.1.2.1.2 Definition Motivation

The above operation definition has been chosen to be particularly useful in the study of linear time invariant systems. In order to see this, consider a linear time invariant system H with unit impulse response h. Given a system input signal x we would like to compute the system output signal $H(x)$. First, we note that the input can be expressed as the convolution

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-k) \tag{3.4}$$

by the sifting property of the unit impulse function. By linearity

$$Hx(n) = \sum_{k=-\infty}^{\infty} x(k)\, H\delta(n-k). \tag{3.5}$$

Since $H\delta(n-k)$ is the shifted unit impulse response $h(n-k)$, this gives the result

$$Hx(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = (x * h)(n). \tag{3.6}$$

Hence, convolution has been defined such that the output of a linear time invariant system is given by the convolution of the system input with the system unit impulse response.

3.5.6.1.2.1.3 Graphical Intuition

It is often helpful to be able to visualize the computation of a convolution in terms of graphical processes. Consider the convolution of two functions f, g given by

$$(f * g)(n) = \sum_{k=-\infty}^{\infty} f(k)\, g(n-k) = \sum_{k=-\infty}^{\infty} f(n-k)\, g(k). \tag{3.7}$$

The first step in graphically understanding the operation of convolution is to plot each of the functions. Next, one of the functions must be selected, and its plot reflected across the $k = 0$ axis. For each real t, that same function must be shifted left by t. The product of the two resulting plots is then constructed. Finally, the area under the resulting curve is computed.

Example 3.1

Recall that the impulse response for a discrete time echoing feedback system with gain a is

$$h(n) = a^n u(n), \tag{3.8}$$

and consider the response to an input signal that is another exponential

$$x(n) = b^n u(n). \tag{3.9}$$

We know that the output for this input is given by the convolution of the impulse response with the input signal

$$y(n) = x(n) * h(n). \tag{3.10}$$

We would like to compute this operation by beginning in a way that minimizes the algebraic complexity of the expression. However, in this case, each possible coice is equally simple. Thus, we would like to compute

$$y(n) = \sum_{k=-\infty}^{\infty} a^k u(k) b^{n-k} u(n-k). \tag{3.11}$$

The step functions can be used to further simplify this sum. Therefore,

$$y(n) = 0 \tag{3.12}$$

for $n < 0$ and

$$y(n) = \sum_{k=0}^{n} (ab)^k \tag{3.13}$$

for $n \geq 0$. Hence, provided $ab \neq 1$, we have that

$$y(n) = \{ \begin{array}{cc} 0 & n < 0 \\ \frac{1-(ab)^{n+1}}{1-(ab)} & n \geq 0 \end{array}. \tag{3.14}$$

3.5.6.1.2.2 Circular Convolution

Discrete time circular convolution is an operation on two finite length or periodic discrete time signals defined by the integral

$$(f * g)(n) = \sum_{k=0}^{N-1} \hat{f}(k) \hat{g}(n-k) \tag{3.15}$$

for all signals f, g defined on $\mathbb{Z}[0, N-1]$ where $\hat{f}, \hat{g}$ are periodic extensions of f and g. It is important to note that the operation of circular convolution is commutative, meaning that

$$f * g = g * f \tag{3.16}$$

for all signals f, g defined on $\mathbb{Z}[0, N-1]$. Thus, the circular convolution operation could have been just as easily stated using the equivalent definition

$$(f * g)(n) = \sum_{k=0}^{N-1} \hat{f}(n-k) \hat{g}(k) \tag{3.17}$$

for all signals f, g defined on $\mathbb{Z}[0, N-1]$ where $\hat{f}, \hat{g}$ are periodic extensions of f and g. Circular convolution has several other important properties not listed here but explained and derived in a later module.

Alternatively, discrete time circular convolution can be expressed as the sum of two summations given by

$$(f * g)(n) = \sum_{k=0}^{n} f(k) g(n-k) + \sum_{k=n+1}^{N-1} f(k) g(n-k+N) \tag{3.18}$$

for all signals f, g defined on $\mathbb{Z}[0, N-1]$.

Meaningful examples of computing discrete time circular convolutions in the time domain would involve complicated algebraic manipulations dealing with the wrap around behavior, which would ultimately be more confusing than helpful. Thus, none will be provided in this section. Of course, example computations in the time domain are easy to program and demonstrate. However, disrete time circular convolutions are more easily computed using frequency domain tools as will be shown in the discrete time Fourier series section.

3.5.6.1.2.2.1 Definition Motivation

The above operation definition has been chosen to be particularly useful in the study of linear time invariant systems. In order to see this, consider a linear time invariant system H with unit impulse response h. Given a finite or periodic system input signal x we would like to compute the system output signal $H(x)$. First, we note that the input can be expressed as the circular convolution

$$x(n) = \sum_{k=0}^{N-1} \hat{x}(k) \hat{\delta}(n-k) \tag{3.19}$$

by the sifting property of the unit impulse function. By linearity,

$$Hx(n) = \sum_{k=0}^{N-1} \hat{x}(k) H \hat{\delta}(n-k). \tag{3.20}$$

Since $H\delta(n-k)$ is the shifted unit impulse response $h(n-k)$, this gives the result

$$Hx(n) = \sum_{k=0}^{N-1} \hat{x}(k) \hat{h}(n-k) = (x * h)(n). \tag{3.21}$$

Hence, circular convolution has been defined such that the output of a linear time invariant system is given by the convolution of the system input with the system unit impulse response.

3.5.6.1.2.2.2 Graphical Intuition

It is often helpful to be able to visualize the computation of a circular convolution in terms of graphical processes. Consider the circular convolution of two finite length functions f, g given by

$$(f * g)(n) = \sum_{k=0}^{N-1} \hat{f}(k) \hat{g}(n-k) = \sum_{k=0}^{N-1} \hat{f}(n-k) \hat{g}(k). \tag{3.22}$$

The first step in graphically understanding the operation of convolution is to plot each of the periodic extensions of the functions. Next, one of the functions must be selected, and its plot reflected across the $k = 0$ axis. For each $k \in \mathbb{Z}[0, N-1]$, that same function must be shifted left by k. The product of the two resulting plots is then constructed. Finally, the area under the resulting curve on $\mathbb{Z}[0, N-1]$ is computed.

3.5.6.1.3 Interactive Element

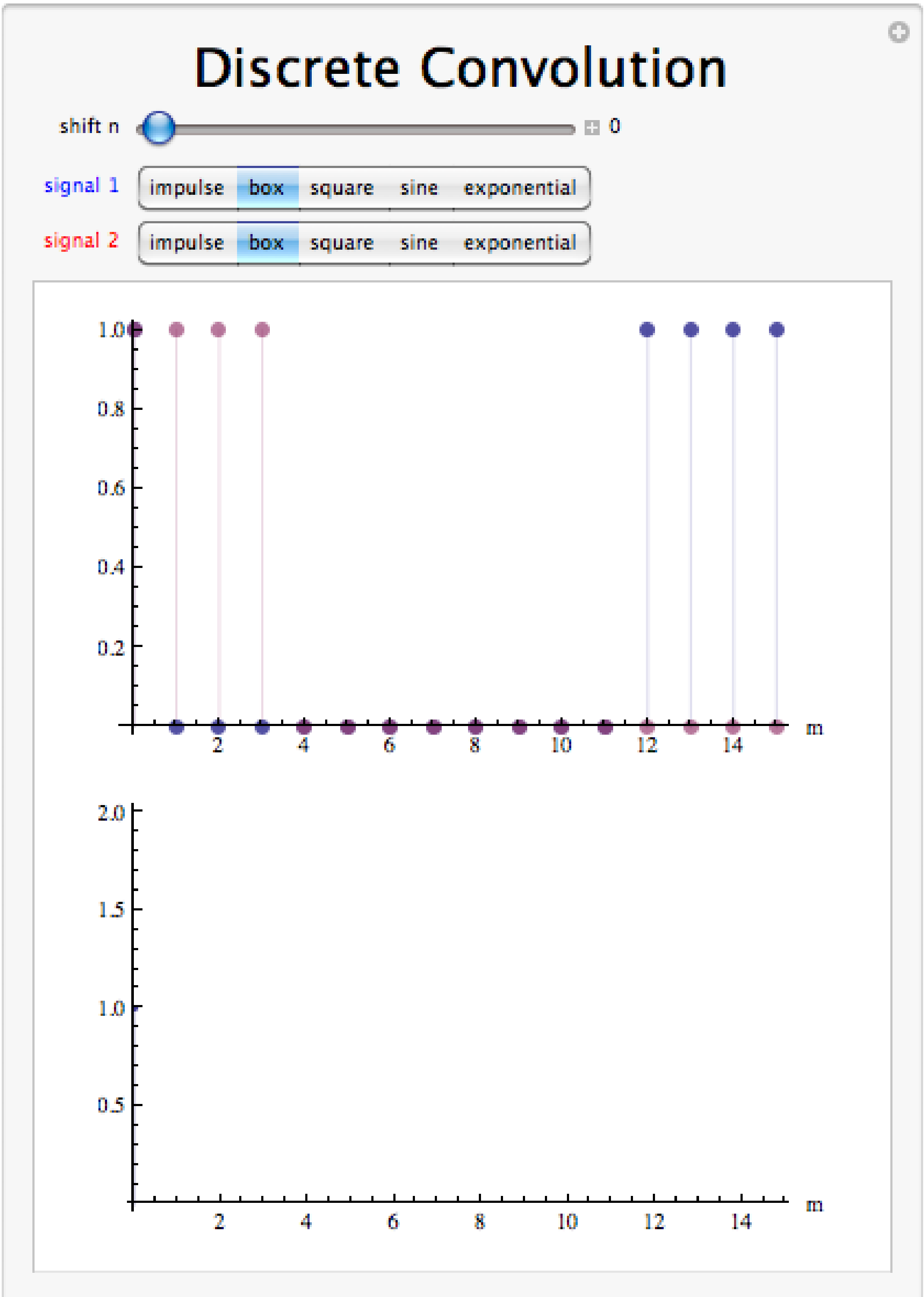


Figure 3.1: Interact (when online) with the Mathematica CDF demonstrating Discrete Linear Convolution. To download, right click and save file as .cdf

3.5.6.1.4 Convolution Summary

Convolution, one of the most important concepts in electrical engineering, can be used to determine the output signal of a linear time invariant system for a given input signal with knowledge of the system's unit impulse response. The operation of discrete time convolution is defined such that it performs this function for infinite length discrete time signals and systems. The operation of discrete time circular convolution is defined such that it performs this function for finite length and periodic discrete time signals. In each case, the output of the system is the convolution or circular convolution of the input signal with the unit impulse response.

Appendix

4.1 Complex numbers and Arithmetic[1]

[1]This content is available online at <http://cnx.org/content/m11497/1.1/>.

4.2 Riemann Integration[2]

[2]This content is available online at <http://cnx.org/content/m11511/1.2/>.

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