Robust Investment Decisions: Models and Solution Approaches

Dr. Nalan Gulpinar

ORMS Group, Warwick Business School, The University of Warwick

IBM Optimisation Forum for Financial Services

Outline

- decision making problems under uncertainty
- robust optimisation formulations
- discrete – single and multi-period scenario based approach
- continuous – uncertainty sets integrated approach
- applications
  - fund management – benchmark tracking
  - portfolio allocation – regime switching model under disruption
  - asset liability management
- concluding remarks

Uncertainty modelling

Traditional approaches

- Sensitivity analysis
  - solve the problem with fixed value of uncertain parameter
  - then investigate sensitivity of the solution to variations of the parameter

- Stochastic programming
  - develop a distributional model for uncertainty
  - generate various sample realizations and
  - solve the problem with expected values of uncertain parameter

Decision-making under uncertainty (SP)

- models and integrates future uncertainty into mathematical programming
- makes optimal decisions to hedge against future good/bad outcomes
- minimizes risk exposure
- uses techniques: scenario based, expected value, multi-criteria optimisation

$P:$ corresponding probability distribution

$$z = \min_x E_y[f(x,y)] = \int f(x,y)\,dP(y)$$

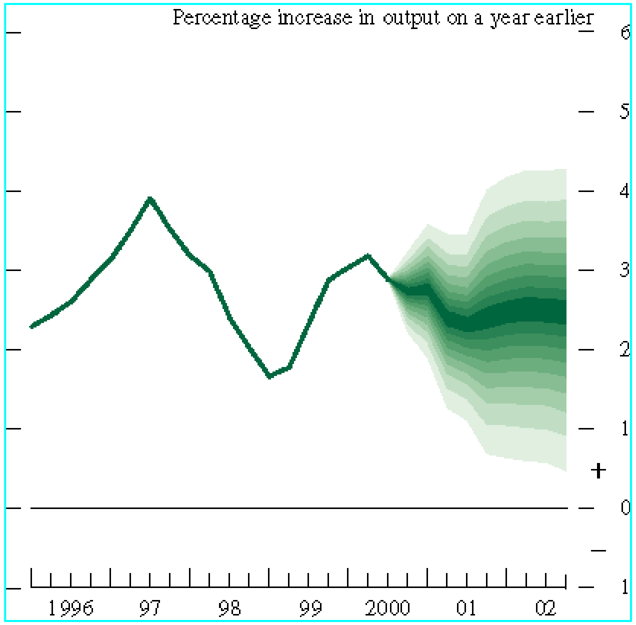
$$\text{s.t} \quad x \in C \subseteq R^n$$

$\Omega:$ discrete, finite set of possible realizations y

$$z = \min_x E_y[f(x,y)] = \sum_{y \in \Omega} f(x,y)\,p(y)$$

$$\text{s.t} \quad x \in C \subseteq R^n$$

Challenging issues in SP



- how to describe randomness?
  - the future using discretised probabilistic model (how to generate scenario tree)
- inaccuracy on data & scenarios (estimation and forecasting errors)
  - no unique scenario tree (different view of the future)
  - how to hedge the risk of making decision on the wrong scenario
- how to handle the size of the problem?
  - number of time periods and number of scenarios
  - decomposition, scenario aggregation
- Robust optimisation

Modelling a stochastic system

- Stochastic program (expected value approach)

$$\min_{x} \; E_y(f(x,y)), \quad y \sim N(\varpi, \Lambda)$$

- Robust optimisation (worst-case approach)

$$f(x^*, y^*) = \min_{x} \max_{y \in R} \; f(x,y)$$

Robustness of mmx:

$$f(x^*, y^*) \geq f(x^*, y), \quad \forall y \in R$$

Expected performance is guaranteed to be at the worst-case, but improves if any scenario other than the worst-case is realised.

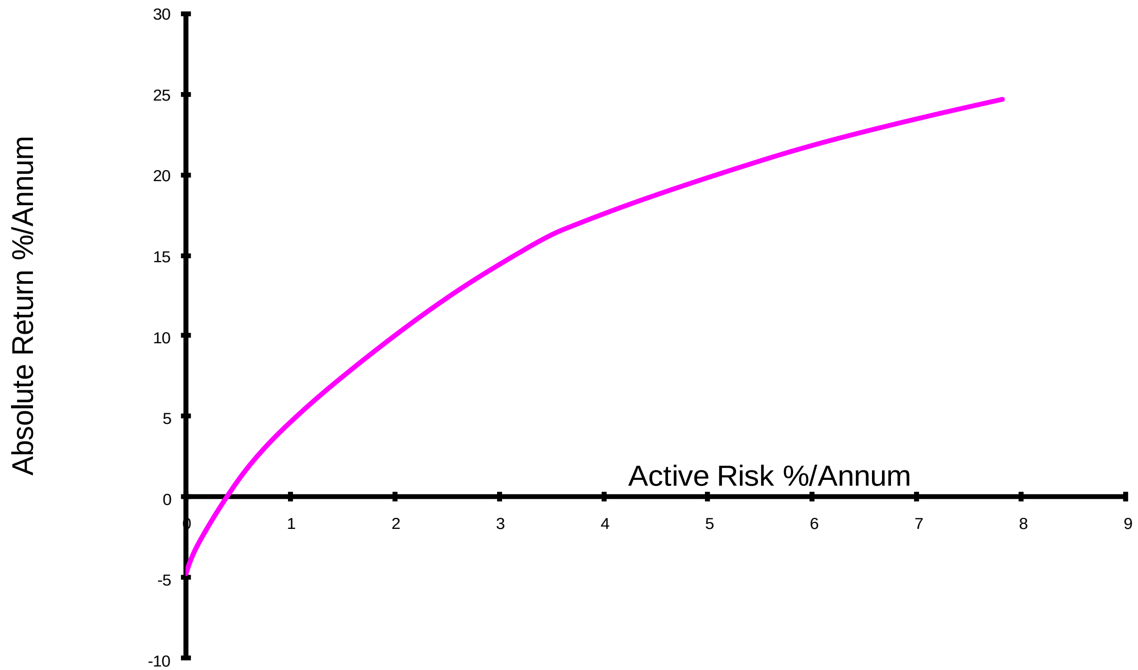
Why worst-case analysis?

- inherently inaccurate random variable forecasts & estimates
- when predicting the future, not possible to settle on single scenario
- rival representation of future in terms of rival scenarios
- proposed method based on the mmx strategy
  - robustness of mmx strategy
  - provides guaranteed performance under worst-case conditions
  - computes optimal decision simultaneously with worst-case
  - takes into account of all rival scenarios rather than single one
  - guards against making decision on a wrong scenario

Single risk-return frontier: Markowitz

$$\max \quad \alpha \boxed{\begin{matrix} expected \\ return \end{matrix}} - (1-\alpha) \boxed{\begin{matrix} expected \\ risk \end{matrix}}$$

$$\max \left\{ \alpha \mathbf{r}^T (\mathbf{w} - \overline{\mathbf{w}}) - (1-\alpha)(\mathbf{w} - \overline{\mathbf{w}})^T \Lambda (\mathbf{w} - \overline{\mathbf{w}}) \; | \mathbf{w} \in V \right\}$$

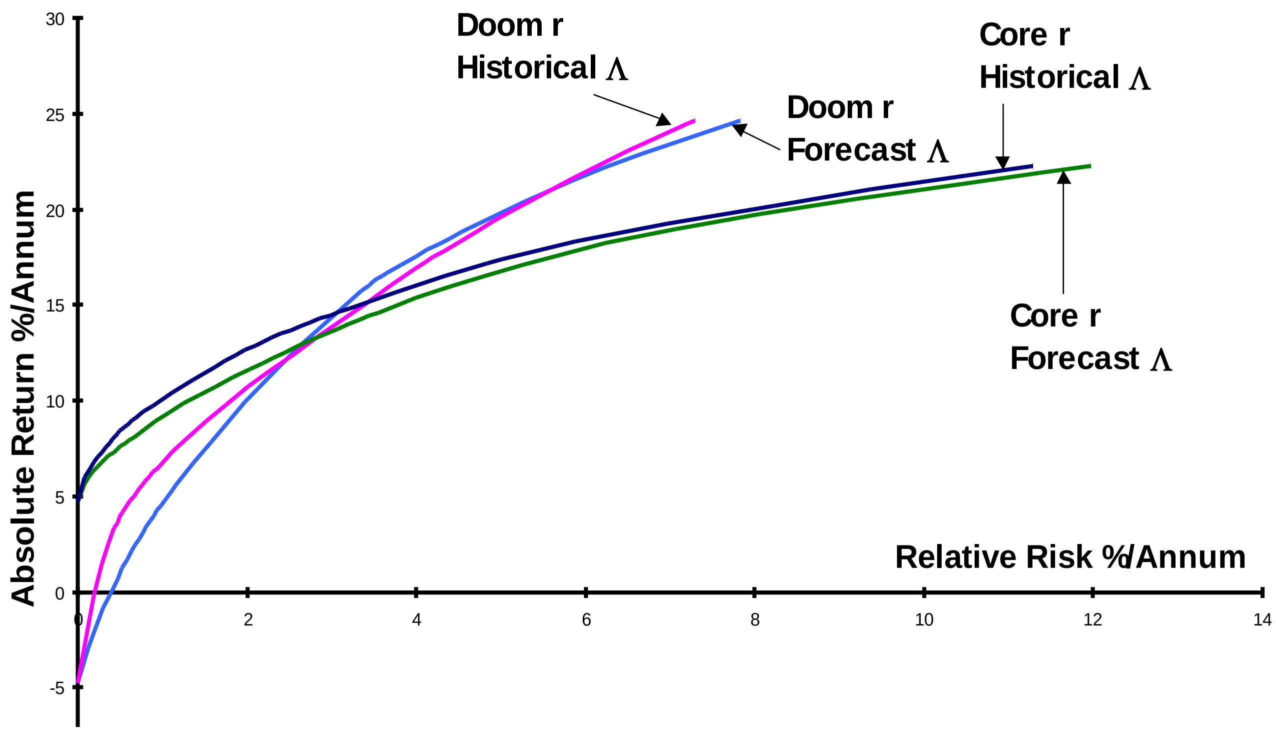


Given risk-aversion value, the optimal investment strategy relative to benchmark portfolio is computed !

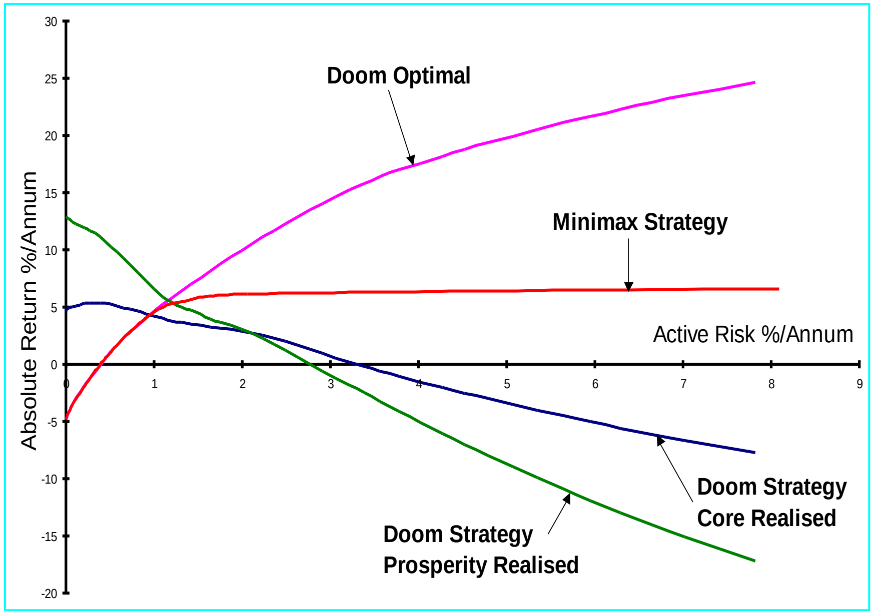
Single risk-return frontier: Markowitz

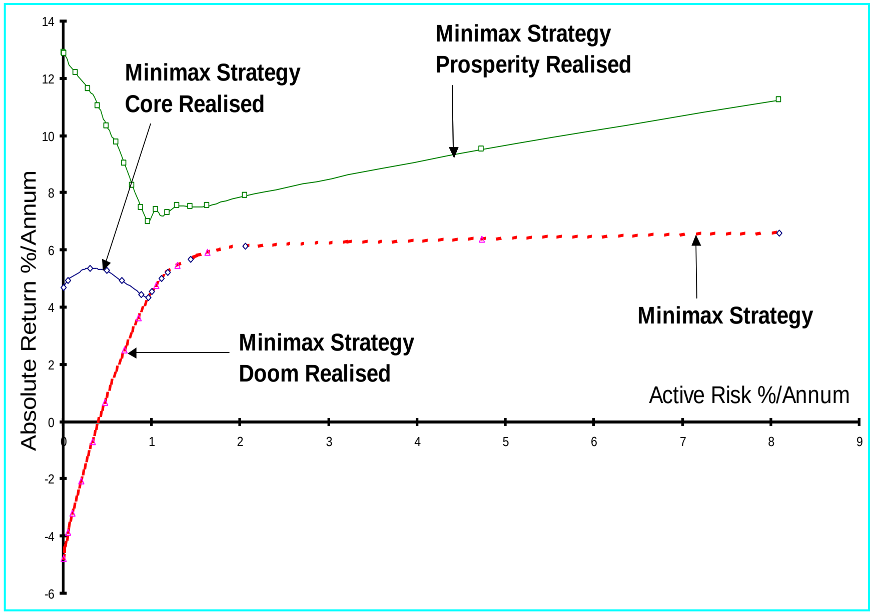
$$\max \quad \alpha \boxed{\text{expected return}} - (1-\alpha) \boxed{\text{expected risk}}$$

$$\max_{\mathbf{w}\in\Omega} \left\{ \alpha\, \mathbf{r}_i^T (\mathbf{w}-\overline{\mathbf{w}}) - (1-\alpha)(\mathbf{w}-\overline{\mathbf{w}})^T \Lambda_j (\mathbf{w}-\overline{\mathbf{w}}) \right\}$$



Single scenario & mmx based optimal strategy





- rival return scenarios: doom, prosperity, core
- m-v frontier for each individual scenario
- evaluate performance of portfolio strategies if any other scenarios are realised
- basic guaranteed performance represented by mmx lower bound

Generalised discrete mmx

- rival return scenarios $r_i \quad i = 1,2,\cdots,I$
- risk forecast $\Lambda_j \quad j = 1,2,\cdots,J$
- benchmark $\overline{w}_k \quad k = 1,2,\cdots,K$
- risk parameter α
- current portfolio position p
- buy-sell costs c_b, c_s

$$\min_{\mathbf{w}\in V} \left\{ \alpha \max_{\substack{j=1,\dots,J \\ k=1,\dots,K}} \left[(\mathbf{w}-\overline{\mathbf{w}}_k)^T \Lambda_j (\mathbf{w}-\overline{\mathbf{w}}_k) \right] - \min_{\substack{i=1,\cdots,I \\ k=1,\cdots,K}} \left[\mathbf{r}_i^T (\mathbf{w}-\overline{\mathbf{w}}_k) - \tau(\mathbf{w},\mathbf{p},\mathbf{c}_{b,s}) \right] \right\}$$

Nonlinear programming formulation

$$\min_{\mathbf{w},\mathbf{b},\mathbf{s}} \; \alpha\upsilon - \mu + \gamma\,\mathbf{b}^T\mathbf{s}$$ weighted return vs. risk

subject to

$$\mathbf{r}_i^T(\mathbf{w} - \overline{\mathbf{w}}_k) - \tau \geq \boldsymbol{\mu}, \qquad \forall \boldsymbol{i}, \boldsymbol{k}$$ μ = worst-case return

$$(\mathbf{w} - \overline{\mathbf{w}}_k)^T \boldsymbol{\Lambda}_j (\mathbf{w} - \overline{\mathbf{w}}_k) \leq \upsilon, \qquad \forall \boldsymbol{j}, \boldsymbol{k}$$ ν = worst-case risk

$$\mathbf{p} + \mathrm{b} - \mathbf{s} = \mathbf{w}$$ transaction volumes

$$\mathbf{c}_b^T\mathbf{b} + \mathbf{c}_s^T\mathbf{s} = \boldsymbol{\tau}$$ τ = transaction costs

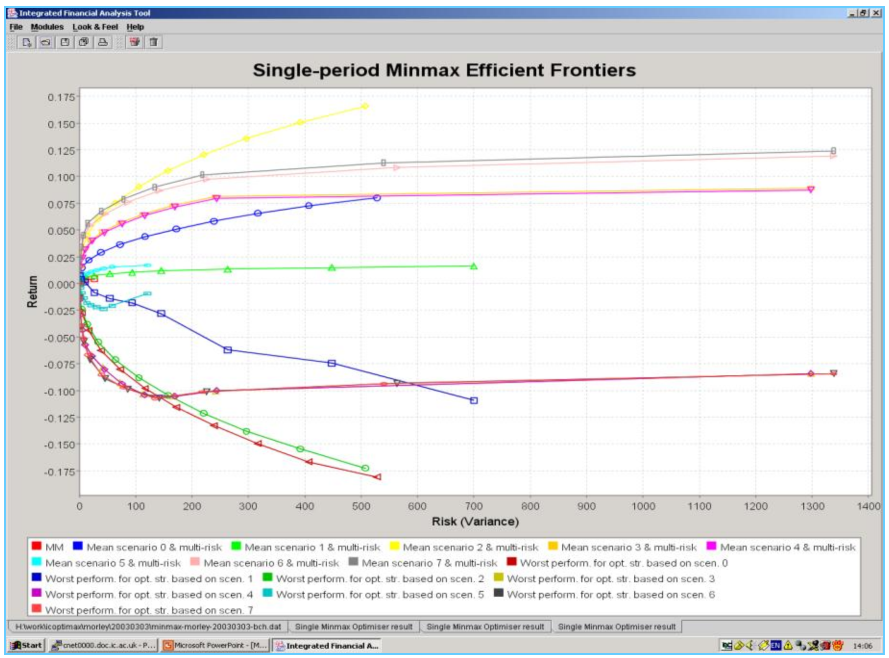
$$\sum_{i=1}^{\mathbf{n}} w_i = 1$$ scaled balance

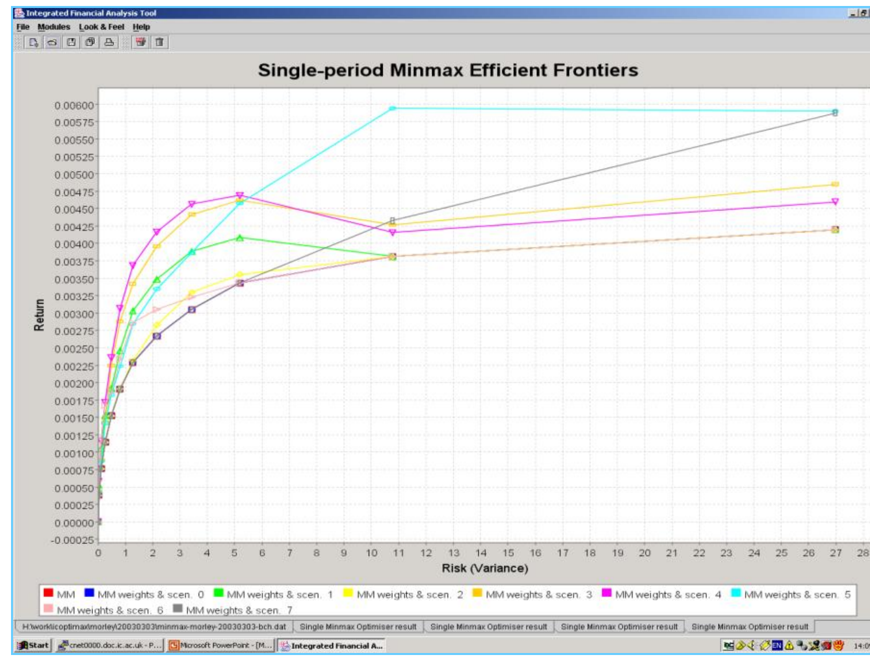
$$\mathbf{w}, \mathbf{b}, \mathbf{s} \geq \mathbf{0}$$

Application 1: fund management

- multi return-risk mmx
- 11 assets
- 8 rival return scenarios
- 10 rival risk scenarios

Scenario 0 =	Low Growth - No War
Scenario 1 =	Low Growth - Clean War
Scenario 2 =	Low Growth - Messy War
Scenario 3 =	Medium Growth - No War
Scenario 4 =	Medium Growth - Clean War
Scenario 5 =	Medium Growth - Messy War
Scenario 6 =	High Growth - No War
Scenario 7 =	High Growth - Clean War





Realisation of the worst-case scenario: mmx vs single scenario optimisation

Robust mmx strategy – guaranteed lower bound of the mmx strategy

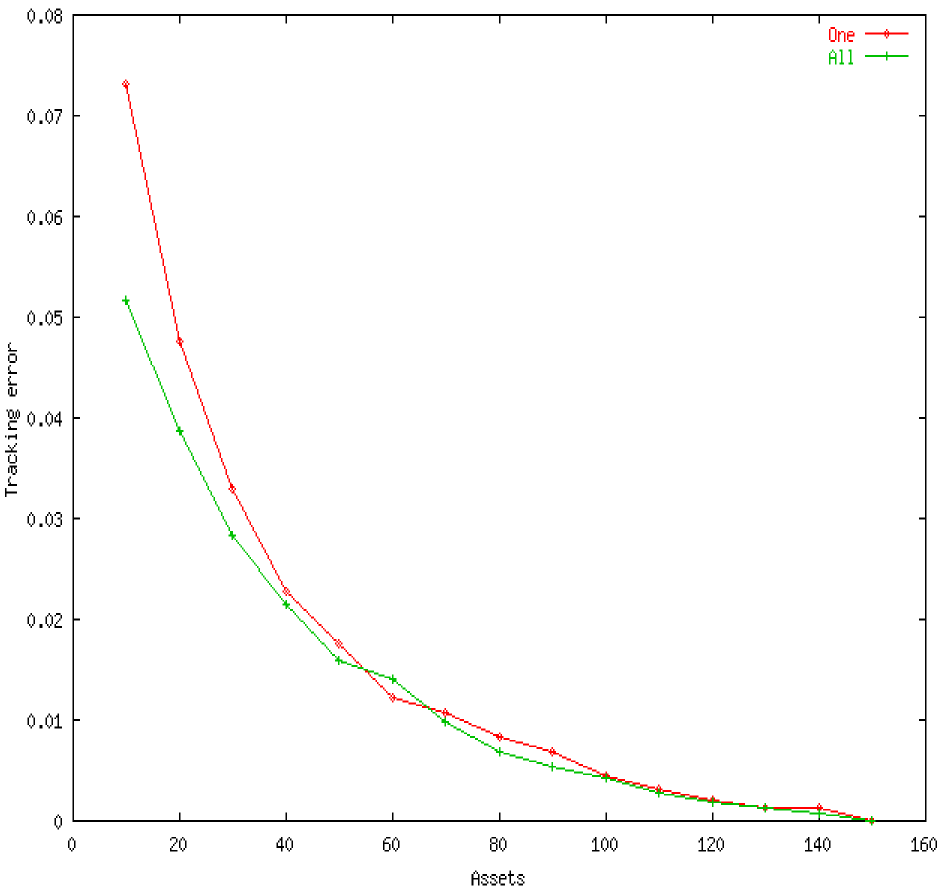
Application 2: robust benchmark tracking

minimisation of tracking error in view of rival risk scenarios

$$\min_{\substack{\sum w_i = 1 \\ \mathbf{w} \geq 0}} \max_{j=1,\cdots,J} \left\{ (\mathbf{w} - \overline{\mathbf{w}})^T \Lambda_j (\mathbf{w} - \overline{\mathbf{w}}) \right\}$$

Empirical example:

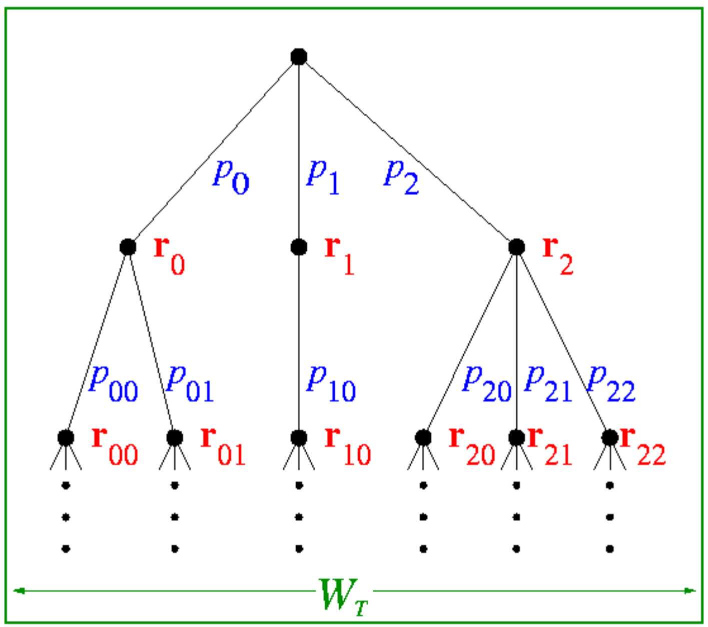
- n = 150 assets, with price history
- I = 0 return scenario
- J = 1 risk scenario (control)
- J = 10 risk scenarios (experiment)
- K = 1 benchmark

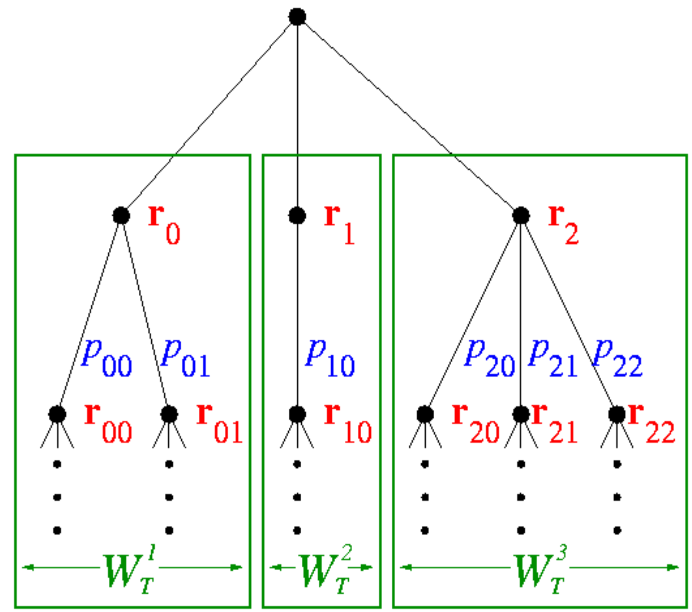


- multiple risk scenarios allow lower error
- the smaller number of assets, greater reduction in tracking error

Multi-period portfolio optimisation

- after initial investment, portfolio at t is restructured in terms of return & risk, and redeemed at T.
- conflicting objectives, benchmark relative, transaction costs, so on





$$\max \left\{\alpha \, \mathrm{E}[W_T] - (1-\alpha)\mathrm{var}[W_T]\right\}$$

$$\max \min_{i,j} \left\{ \alpha \mathrm{E}\left[W_T^i\right] - (1-\alpha)\mathrm{var}\left[W_T^j\right] \right\}$$

Multi-period mmx portfolio strategy

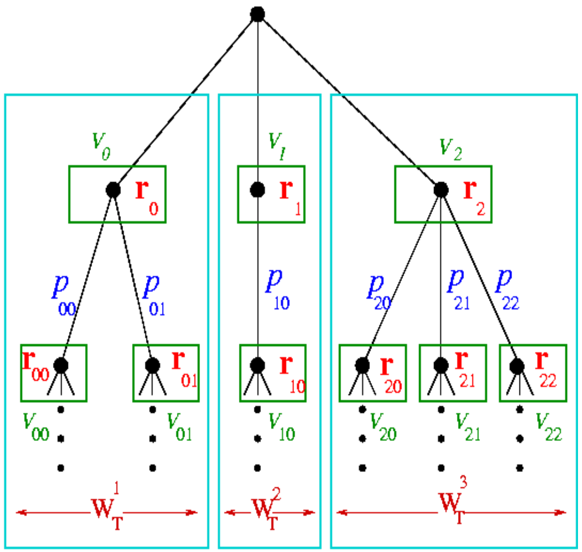
❖ ***i*** covariance matrices at each node of scenario tree & ***k*** rival return scenarios

$$\min_{w}\left\{\gamma\sum_{t=1}^{T}\alpha_t\sum_{e\in N_t}\max_{i}\left[P_e\left(w_{a(e)}-\overline{w}_{a(e)}\right)'\left(\Lambda_i+r_e{}'r_e\right)\left(w_{a(e)}-\overline{w}_{a(e)}\right)\right]-\min_{k}\left[\sum_{e\in N_T{}^k}P_e\left(w_{a(e)}-\overline{w}_{a(e)}\right)'r_e\right]\right\}$$

$$i=1,\cdots,I_e,\quad k=1,\cdots,K,\ t=1,\cdots,T,\ e\in N_t$$

➢ worst-case wealth at each sub-tree

$$\sum_{e\in N_T{}^k}P_e\left(w_{a(e)}-\overline{w}_{a(e)}\right)'r_e\geq\mu\quad e\in N_I,\quad k=1,\cdots,K$$



➢ worst-case risk at node of scenario tree

$$P_e\left(w_{a(e)}-\overline{w}_{a(e)}\right)'\left(\Lambda_i+r_e{}'r_e\right)\left(w_{a(e)}-\overline{w}_{a(e)}\right)\leq v_e\quad e\in N_I,\ i=1,\cdots,I_e$$

Robust strategies using uncertainty sets

- Robust decision problems permit the determination of worst-case optimal decisions given uncertainty sets around the random parameters.

$$\max \{\mathbf{c}'\mathbf{x} | f(\mathbf{x}, \tilde{\mathbf{z}}) \leq 0, \ \ \mathbf{x} \in V\}$$

its robust counterpart

$$\max_{\mathbf{x}} \ \min_{\mathbf{z}} \{\mathbf{c}'\mathbf{x} | f(\mathbf{x}, \mathbf{z}) \leq 0, \ \ \forall\, \mathbf{z} \in \mathcal{U}(\tilde{\mathbf{z}}), \ \ \mathbf{x} \in V\}$$

a tractable optimisation problem with no random parameter

- incorporates data uncertainties into a deterministic framework
- explicitly considers estimation error within the optimization process
- developed independently by Ben-Tal and Nemirovski

Literature review

- an extensive literature in the subject RO for portfolio management
  - A. Ben-Tal and A.S. Nemirovski, "Robust convex optimization", Math. Operations Research, 1998
  - A. Ben-Tal and A.S. Nemirovski, "Robust solutions to uncertain linear programs", OR Letters, 1999
  - L. El Ghaoui, F. Oustry, and H. Lebret, "Robust solutions to uncertain semi-definite programs", 1999
  - M. Lobo and S. Boyd, “The worst -case risk of a portfolio”, 1999
  - A. Ben-Tal, T. Margalit, A. Nemirovski, “Robust modeling of multi-stage portfolio problems”, 2000
  - R. Tütüncü, M. Koenig, “Robust asset allocation”, 2002
  - D. Goldfarb, G. Iyengar, “Robust portfolio selection problems”, Math of OR, 2003
  - L. Garlappi, R. Uppal, T. Wang, “Portfolio selection with parameter &model uncertainty: A Multi-Prior Approach” 2004
  - D. Bertsimas, M. Sim, “Robust discrete optimization and downside risk measures”, 2005
  - S. Ceria , R. Stubbs, “Incorporating estimation errors into portfolio selection: Robust portfolio construction”, 2006
  - N.Gulpinar, B.Rustem, “Robust optimal decisions with imprecise forecasts”, Comp. Statistics *& Data Analysis*, 2007
  - N. Gulpinar, B. Rustem, “Worst-case robust decisions for multi-period portfolio optimization”, EJOR, 2007
  - D. Bertsimas, D. Pachamanova, “Robust multi-period portfolio management in the presence of transaction costs”, Computers & Operations Research, 2008
  - N. Gulpinar D. Pachamanavo, K. Katata, “Robust MV with discrete asset constraints”, J. of Asset Management, 2011.

Asset allocation models

- based on several assumptions on the underlying price dynamics
- performance depends on how accurately the random nature of asset prices is captured
- statistical measurements do not unfold the complete dynamics of the market
- inherently involve estimation errors (imprecise forecasts)
- Robust Optimisation addresses data uncertainty from the perspective of computational tractability

Modeling oil prices under supply disruption

Geometric mean reversion process for stock prices is given by the stochastic differential equation

$$dS(t) = \alpha\left(\mu - \ln S(t)\right) S(t)dt + \sigma S(t)dZ_t$$

and can be written as

$$S(t) = \left(S(0)\right)^{e^{-\alpha t}} \exp\left(\mu\left(1 - e^{-\alpha t}\right)\right) \exp\left(\frac{\sigma}{\sqrt{2\alpha}}\sqrt{\left(1 - e^{-2\alpha t}\right)}Z_t\right)$$

Mean reverting with jump process for stock prices under supply disruption

$$S(t) = \left(S(0)\right)^{e^{-\alpha t}} \exp\left(\mu\left(1 - e^{-\alpha t}\right)\right) \exp\left(\frac{\sigma}{\sqrt{2\alpha}}\sqrt{\left(1 - e^{-2\alpha t}\right)}Z\right) + \int_0^t J(t)e^{-\alpha(t-s)}dQ_s$$

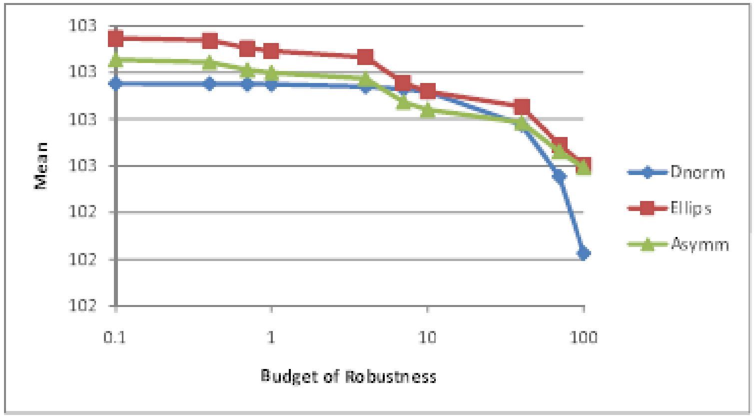
Stock price follows discrete jump process

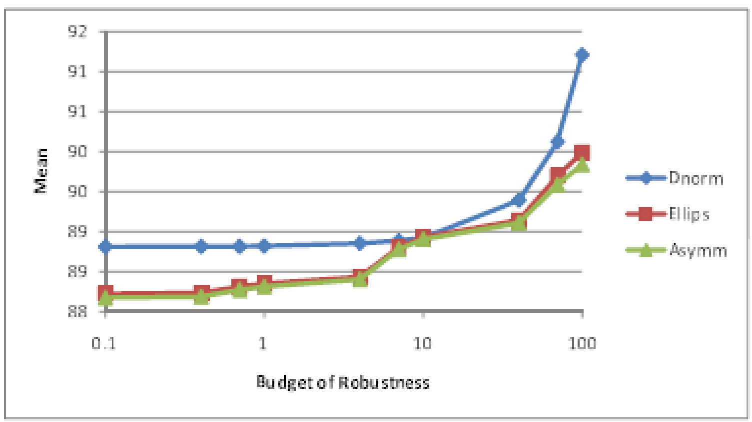
$$S_i(t) = \bar{S}_i(t)\exp\left(\theta_i(t)\,\tilde{z}(t)\right) + \sum_{k\in K_i(t)} \tilde{y}_k^i e^{-\alpha_i(t-t_k)}$$

where $\bar{S}_i(t) = \left(S_i(0)\right)^{e^{-\alpha_i t}} \exp\left(\mu_i\left(1 - e^{-\alpha_i t}\right)\right)$, and $\theta_i(t) = \frac{\sigma}{\sqrt{2\alpha_i}}\sqrt{1 - e^{-2\alpha_i t}}$.

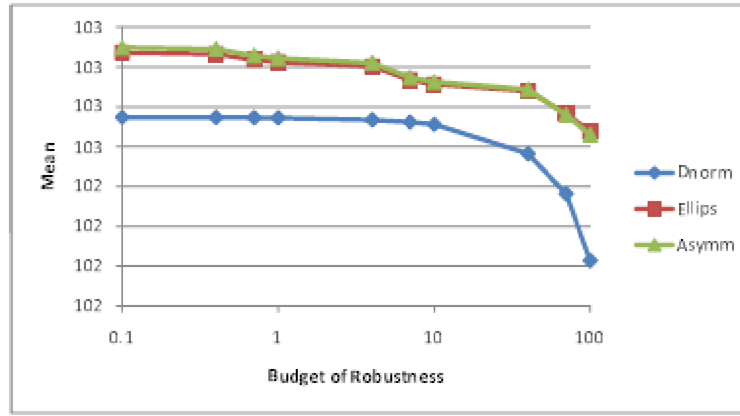
Impact of price of robustness and uncertainty sets

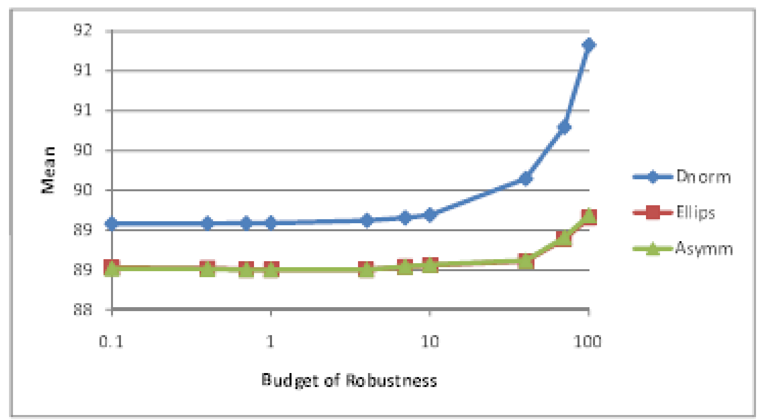
No disruption state





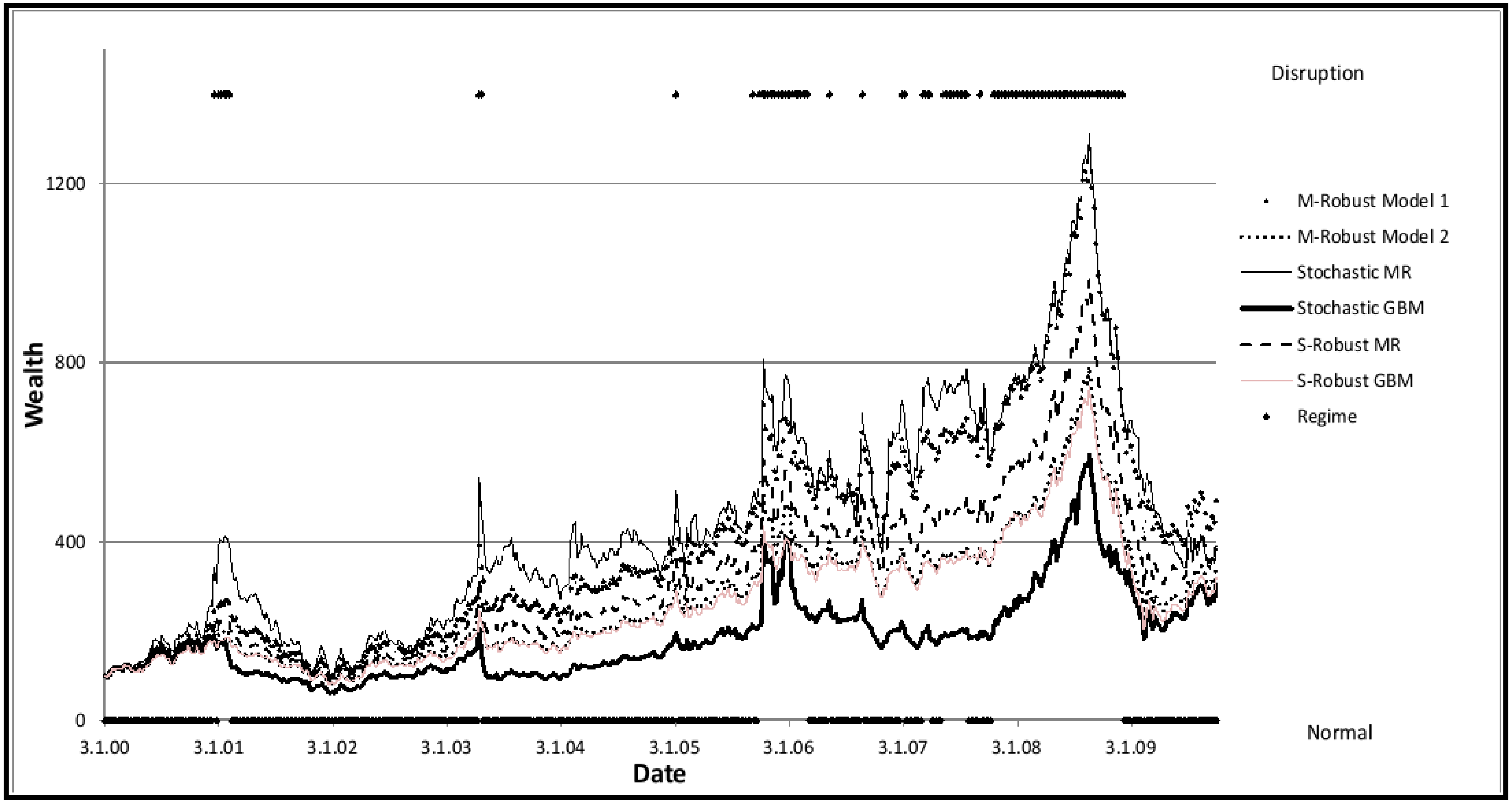
Disruption state





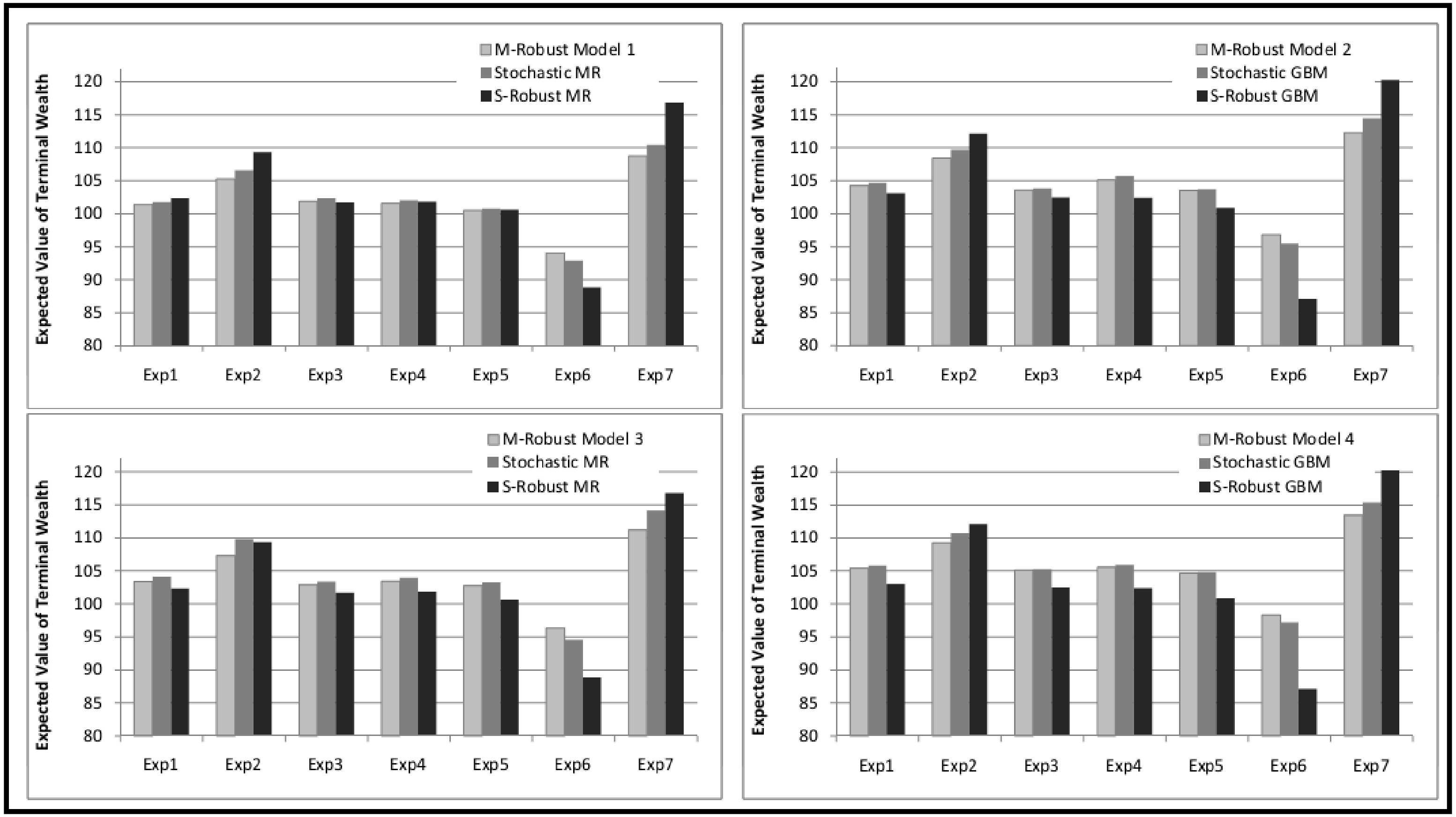
Future prices are realised according to expectations or worse than expected

Performance comparison of strategies



- ❖ SP shows more volatile progress while robust strategies are more conservative; in particular, at disruption state of the market.
- ❖ the multi-regime models outperform to the single-regime ones.

Performance comparison of strategies



- SP produces higher wealth than RO regardless choice of stochastic price processes.
- RO outperform in catastrophic situations (Exp 6 - low returns on commodities)
- Single regime portfolio strategy outperforms to multi-regime (Exps 2 & 7)

Asset Liability Management (ALM)

- SP has been successfully applied in some instances of pension funds

  (e.g. Gondzio &Kouwenberg (2001), Mulvey, Consiglio et al. (2008), Escudero et al. (2009))

- It is still found difficult to use in practice for several reasons
  - large problem size and computational difficulty to solve
  - scenario generation requires sophisticated statistical techniques
  - unknown data about the specific distributions of future uncertainties

  In many cases, general information about the uncertainties (means, ranges, and deviations) may be preferable rather than generating specific scenarios

- Robust optimisation is an alternative approach
  - based on worst-case analysis and computationally tractable

ALM model for pension funds

- A typical pension fund
  - collects premiums from sponsors/currently active employees
  - pays pensions to retired employees, and also invests available funds
- The fund aims to
  - manage assets so that at each time period total value of all assets exceeds company’s future liabilities.
  - at the same time, minimize the contribution rate by the sponsor/active employees of the fund.
- The stochastic ALM problem determines
  - optimal contribution rate and
  - investment strategy during an investment horizon.

Design of computational experiments

Description	Parameters
Time period (T)	6
Number of stocks (M)	10
Transaction costs	2%
Liabilities	[10,20]
Contribution of wages at most	12%
Interest rates	[0.01, 0.05]
Number of factors	5

❖ The forward and backward deviations for factor random variables are computed by the procedure described by Natarajan et al. (2008) using a series of simulations.

❖ Simulate a number of scenarios for cumulative returns (based on lognormal factor model) and evaluate terminal wealth for optimal strategies.

❖ all simulation parameters are selected as in Ben-Tal, Margalit, Nemirovski (2000)

Change in the market

symmetric uncertainty set

Models	mean	variance	VaR	CVaR	min	max
Nominal	11.25	395.72	-15.23	-20.26	-32.79	113.64
0.1	8.31	125.84	-8.20	-10.84	-20.58	75.15
0.3	1.98	26.40	-5.85	-7.11	-10.68	31.98
0.5	-3.17	9.49	-7.79	-8.69	-11.16	14.26
0.7	-8.25	2.17	-10.53	-11.06	-12.20	-0.59
1	-15.39	0.60	-16.70	-16.98	-17.70	-13.08

Models	mean	variance	VaR	CVaR	min	max
Nominal	-19.26	156.81	-36.10	-38.01	-42.56	50.20
0.1	-20.68	39.03	-29.35	-31.09	-36.04	7.22
0.3	-16.59	7.77	-20.75	-21.64	-23.77	-5.64
0.5	-16.81	3.13	-19.64	-20.17	-21.51	-9.53
0.7	-14.67	1.02	-16.36	-16.74	-17.35	-10.95
1	-15.37	0.61	-16.69	-16.96	-17.54	-13.12

asymmetric uncertainty set

Models	mean	variance	VaR	CVaR	min	max
Nominal	11.25	395.72	-15.23	-20.26	-32.79	113.64
0.1	8.74	169.88	-9.52	-12.46	-20.71	82.05
0.3	3.02	69.05	-6.25	-8.64	-12.09	43.92
0.5	-2.50	86.52	-8.24	-9.82	-12.50	16.96
0.7	-6.27	4.13	-10.63	-11.75	-14.35	20.29
1	-14.84	58.70	-16.53	-17.24	-19.70	9.86

Models	mean	variance	VaR	CVaR	min	max
Nominal	-19.26	156.81	-36.10	-38.01	-42.56	50.20
0.1	-20.57	41.18	-29.66	-31.18	-36.95	12.85
0.3	-16.99	32.96	-22.20	-24.79	-24.74	-4.76
0.5	-17.17	26.39	-22.25	-21.73	-24.86	-1.50
0.7	-15.03	12.46	-19.83	-20.38	-19.54	-9.25
1	-15.56	12.76	-18.41	-18.61	-20.98	-12.27

- Factors generated as zero (top) and negative mean (bottom) and unit covariance
- Nominal strategy provides higher wealth than the robust model
- Robust asymmetric strategy captures asymmetry in lognormal returns better

Summary

- robust investment models using rival scenarios and uncertainty sets
- address data uncertainty in financial applications
  - alternative approach to stochastic program
  - computationally tractable
  - provides a guaranteed performance
- choice of uncertainty sets and price of robustness plays an important role on the performance of investment strategies